Chapter 9

Lecture: The Schwarzschild Spacetime

> One of the simplest solutions to the Einstein equations corresponds to a metric that describes the gravitational field exterior to a static, spherical, uncharged mass without angular momentum and isolated from all other mass (Schwarzschild, 1916).

The Schwarzschild solution is

- A solution to the vacuum Einstein equations $G_{\mu\nu} = R_{\mu\nu} = 0$.
- Only valid in the absence of matter and non-gravitational fields ($T_{\mu\nu} = 0$).
- Spherically symmetric and time independent.

Thus, the Schwarzschild solution is valid outside spherical mass distributions, but the interior of a star will be described by a different metric that must be matched at the surface to the Schwarzschild one.

9.1 The Form of the Metric

Work in spherical coordinates (r, θ, φ) and seek a time-independent solution assuming

- The angular part of the metric will be unchanged from its form in flat space because of the spherical symmetry.
- The parts of the metric describing dt and dr will be modified by functions that depend on the radial coordinate r.

Therefore, let us write the 4-D line element as

$$ds^2 = \underbrace{-B(r)dt^2 + A(r)dr^2}_{\text{Modified from flat space}} \underbrace{+r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2}_{\text{Same as flat space}},$$

where $A(r)$ and $B(r)$ are unknown functions that may depend on r but not time. They may be determined by

1. Requiring that this metric be consistent with the Einstein field equations for $T_{\mu\nu} = 0$.
2. Imposing physical boundary conditions.

Boundary conditions: Far from the star gravity becomes weak so

$$\operatorname*{Lim}_{r\to\infty} A(r) = \operatorname*{Lim}_{r\to\infty} B(r) = 1.$$

Substitute the metric form in vacuum Einstein and impose these boundary conditions (Exercise):

1. With the assumed form of the metric,

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

   compute the non-vanishing connection coefficients $\Gamma^{\lambda}_{\mu\nu}$.

$$\Gamma^{\sigma}_{\lambda\mu} = \tfrac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right)$$

2. Use the connection coefficients to construct the Ricci tensor $R_{\mu\nu}$.

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma},$$

   (Only need $R_{\mu\nu}$, not full $G_{\mu\nu}$ since we will solve vacuum Einstein equations.

3. Solve the coupled set of equations

$$R_{\mu\nu} = 0$$

subject to the boundary conditions

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1.$$

The solution requires some manipulation but is remarkably simple:

$$B(r) = 1 - \frac{2M}{r} \qquad A(r) = B(r)^{-1},$$

where M is the single parameter. The line element is then

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where $d\tau^2 = -ds^2$. The corresponding metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}.$$

which is diagonal but not constant.

By comparing

$$\underbrace{g_{00} = -\left(1 - \frac{2GM}{rc^2}\right)}_{\text{Weak gravity (earlier)}} \longleftrightarrow \underbrace{g_{00} = -\left(1 - \frac{2GM}{rc^2}\right)}_{\text{Schwarzschild (G \& c restored)}}$$

we see that the parameter M (mathematically the single free parameter of the solution) may be identified with the *total mass that is the source of the gravitational curvature:*

- Rest mass
- Contributions from mass–energy densities and pressure
- Energy from spacetime curvature

From the structure of the metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

- θ and φ have similar interpretations as for flat space.
- The *coordinate radius* r generally cannot be interpreted as a physical radius because $A(r) \neq 1$.
- The *coordinate time* t generally cannot be interpreted as a physical clock time because $B(r) \neq 1$.

> The quantity
>
> $$r_{\mathrm{S}} \equiv 2M$$
>
> is called the *Schwarzschild radius.* It plays a central role in the description of the Schwarzschild spacetime.

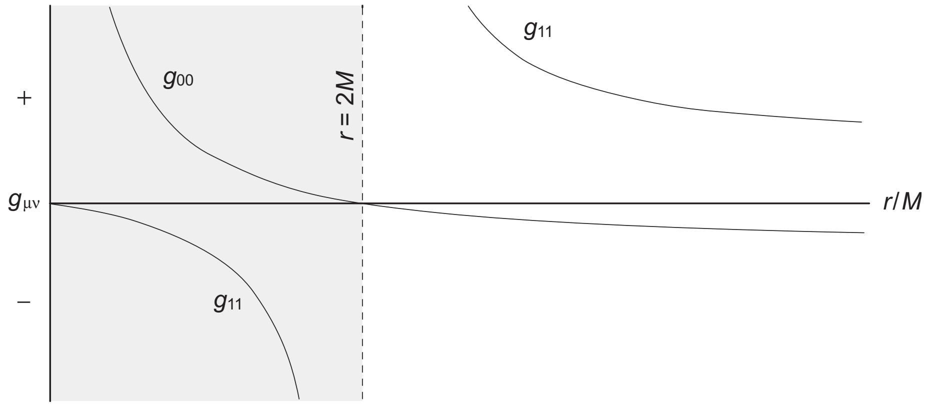

Figure 9.1: The components g_{00} and g_{11} in the Schwarzschild metric.

The line element (metric)

$$ds^2 = \underbrace{-\left(1 - \frac{2M}{r}\right)}_{g_{00}} dt^2 + \underbrace{\left(1 - \frac{2M}{r}\right)^{-1}}_{g_{11}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

appears to contain two singularities (see above figure)

1. A singularity at $r = 0$ from g_{00} (an *essential singularity*).

2. A singularity at $r = r_S = 2M$ from g_{11} (a *coordinate singularity*).

Coordinate Singularity: Place where a chosen set of coordinates does not describe the geometry properly.

> Example: At the North Pole the azimuthal angle φ takes a continuum of values $0–2\pi$, so all those values correspond to a single point. But this has *no physical significance.*

Coordinate singularities are *not essential* and can be removed by a *different choice of coordinate system.*

9.1.1 Measuring Distance and Time

What is the physical meaning of the coordinates (t, r, θ, φ)?

- We may assign a practical definition to the radial coordinate r by

  1. Enclosing the origin of our Schwarzschild spacetime in a series of concentric spheres,
  2. Measuring for each sphere a surface area (conceptually by laying measuring rods end to end),
  3. Assigning a radial coordinate r to that sphere using $\text{Area} = 4\pi r^2$.

- Then we can use distances and trigonometry to define the angular coordinate variables θ and φ.
- Finally we can define coordinate time t in terms of clocks attached to the concentric spheres.

For Newtonian theory with its implicit assumption that events occur on a passive background of euclidean space and constantly flowing time, that's the whole story.

But in curved Schwarzschild spacetime

- The coordinates (t, r, θ, φ) provide a global reference frame for an observer making measurements an infinite distance from the gravitational source of the Schwarzschild spacetime.
- However, physical quantities measured by arbitary observers are not specified directly by these coordinates but rather *physical quantitites must be computed from the metric*.

Proper and Coordinate Distances

Consider distance measured in the radial direction. Set

$$dt = d\theta = d\varphi = 0$$

in the line element to obtain an interval of radial distance

$$ds^2 = \underbrace{-\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2}_{\text{set } t,\theta,\varphi \text{ to constants } \rightarrow\, dt=d\theta=d\varphi=0}$$

$$ds = \frac{dr}{\sqrt{1-\frac{2GM}{rc^2}}},$$

- In this expression we term
  1. ds the *proper distance* and
  2. dr the *coordinate distance.*
- The physical interval in the radial direction measured by a local observer is given by the proper distance ds, *not* by dr.
- GM/rc^2 is a measure of the strength of gravity, so the proper distance and coordinate distance are equivalent only if gravity is negligibly weak, either because
  1. The source M is weak, or
  2. We are a very large coordinate distance r from the source.

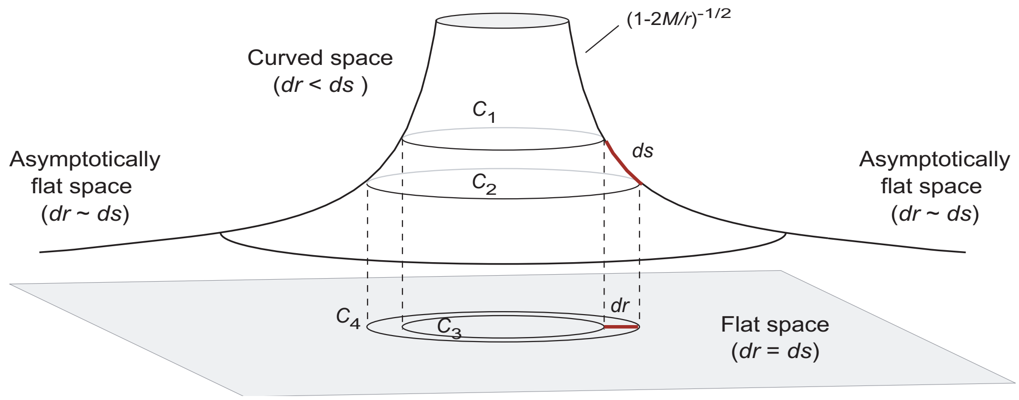

Figure 9.2: Relationship between radial coordinate distance *dr* and proper distance *ds* in Schwarzschild spacetime.

The relationship between the coordinate distance interval *dr* and the proper distance interval *ds* is illustrated further in Fig. 9.2.

- The circles C_1 and C_3 represent spheres having radius r in euclidean space.
- The circles C_2 and C_4 represent spheres having an infinitesimally larger radius $r + dr$ in euclidean space.
- In euclidean space the distance that would be measured between the spheres is *dr*
- But in the curved space the measured distance between the spheres is *ds*, which is larger than *dr*, by virtue of

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}},$$

- Notice however that at large distances from the source of the gravitational field the Schwarzschild spacetime becomes flat and then $dr \sim ds$.

Proper and Coordinate Times

Likewise, to measure a time interval for a stationary clock at r set $dr = d\theta = d\varphi = 0$ in the line element and use $ds^2 = -d\tau^2 c^2$ to obtain

$$ds^2 = \underbrace{-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2}_{\text{set } r,\theta,\varphi \text{ to constants } \rightarrow dr=d\theta=d\varphi=0}$$

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}}\, dt.$$

- In this expression $d\tau$ is termed the *proper time* and dt is termed the *coordinate time*.
- The physical time interval measured by a local observer is given by the proper time $d\tau$, *not* by the coordinate time dt.
- dt and $d\tau$ coincide only if the gravitational field is weak.

Thus we see that for the gravitational field outside a spherical mass distribution

- The coordinates r and t correspond directly to physical distance and time in Newtonian gravity.
- In general relativity the physical (proper) distances and times *must be computed from the metric and are not given directly by the coordinates.*
- Only in regions of spacetime where gravity is very weak do we recover the Newtonian interpretation.

This is as it should be: *The goal of relativity is to make the laws of physics independent of the coordinate system in which they are formulated.*

The coordinates in a physical theory are like street numbers.

- They provide a labeling that locates points in a space, but knowing the street numbers is not sufficient to determine distances.
- We can't answer the question of whether the distance between 36th Street and 37th street is the same as the distance between 40th Street and 41st Street until we know whether the streets are equally spaced.
- We must compute distances from a *metric* that gives a distance-measuring prescription.
  - Streets that are always equally spaced correspond to a "flat" space.
  - Streets with irregular spacing correspond to a position-dependent metric and thus to a "curved" space.

For the flat space the difference in street number corresponds directly (up to a scale) to a physical distance, but in the more general (curved) case it does not.

9.1.2 Embedding Diagrams

> It is sometimes useful to form a mental image of the structure for a curved space by embedding the space or a subset of its dimensions in 3-D euclidean space.

Such embedding diagrams can be misleading, as illustrated well by the case of a cylinder embedded in 3-D euclidean space, which suggests that a cylinder is curved. But it isn't:

- The cylinder is intrinsically a flat 2-D surface: (a) cut it and roll it out into a plane, or (b) calculate its vanishing gaussian curvature.
- The cylinder has *no intrinsic curvature;* the appearance of curvature derives entirely from the embedding in 3-D space and is termed *extrinsic curvature.*

Nevertheless, the image of the cylinder embedded in 3D euclidean space is a useful representation of many properties associated with a cylinder.

We can embed only 2 dimensions of Schwarzschild spacetime in 3D euclidean space.

- Illustrate by choosing $\theta = \pi/2$ and $t = 0$, to give a 2-D metric

$$d\ell^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\varphi^2.$$

- The metric of the 3-D embedding space is conveniently represented in cylindrical coordinates as

$$d\ell^2 = dz^2 + dr^2 + r^2 d\varphi^2$$

- This can be written on $z = z(r)$ as

$$d\ell^2 = \left(\frac{dz}{dr}\right)^2 dr^2 + dr^2 + r^2 d\varphi^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\varphi^2$$

- Comparing

$$d\ell^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\varphi^2 \leftrightarrow d\ell^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\varphi^2$$

  implies that

$$z(r) = 2\sqrt{2M(r - 2M)},$$

  which defines an *embedding surface* $z(r)$ having a geometry that is the same as the Schwarzschild metric in the $(r - \varphi)$ plane.

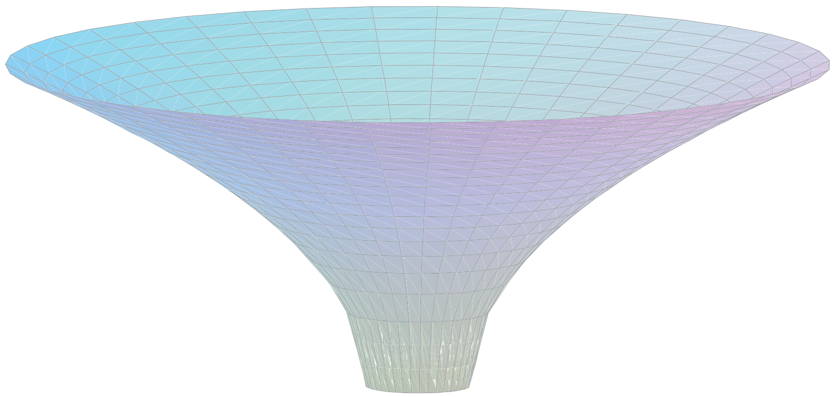

Figure 9.3: An embedding diagram for the Schwarzschild $(r - \varphi)$ plane.

Fig. 9.3 illustrates the embedding function

$$z(r) = 2\sqrt{2M(r - 2M)}$$

Fig. 9.3 is not what a black hole "looks" like, but it is a striking and useful visualization of the Schwarzschild geometry. Thus such embedding diagrams are a standard representation of black holes in popular-level discussion.

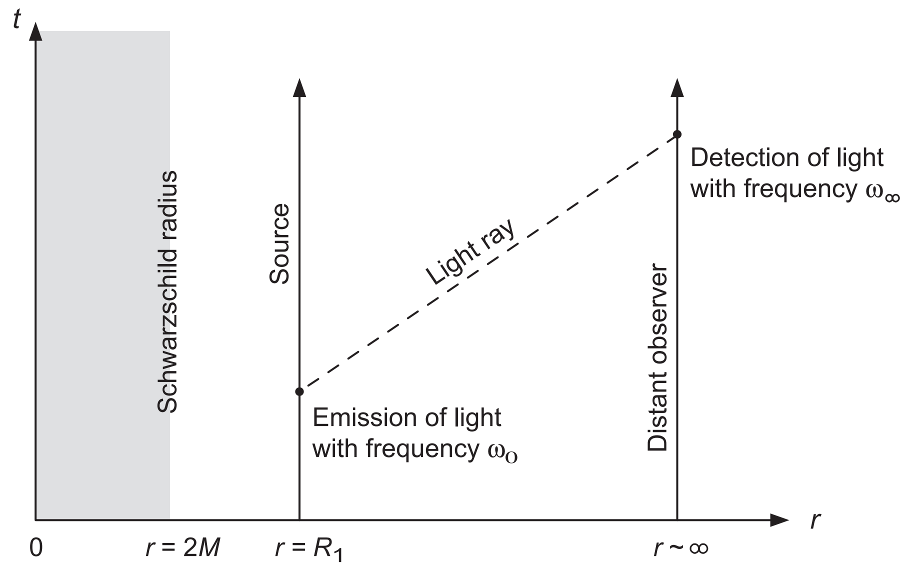

Figure 9.4: A spacetime diagram for gravitational redshift in the Schwarzschild metric.

9.1.3 The Gravitational Redshift

Let's now return to the gravitational redshift problem.

> Emission of light from a radius R_1 that is then detected by a stationary observer at a radius $r >> R_1$ (Fig. 9.4).

For an observer with 4-velocity u, the energy measured for a photon with 4-momentum p is

$$E = \hbar\omega = -p \cdot u,$$

Observers stationary in space but not time so

$$u^i(r) = 0 \qquad u^0 \neq 0$$

Thus the 4-velocity normalization gives

$$u \cdot u = g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \underbrace{g_{00}(x) u^0(r) u^0(r) = -1.}_{\text{Solve for } u^0(r)}$$

and we obtain

$$u^0(r) = \sqrt{\frac{-1}{g_{00}}} = \left(1 - \frac{2M}{r}\right)^{-1/2}.$$

> Symmetry: Schwarzschild metric *independent of time,* which implies the existence of a *Killing vector*
>
> $$\xi^\mu = (t, r, \theta, \varphi) = (1, 0, 0, 0)$$
>
> associated with symmetry under time displacement.

Thus, for a stationary observer at a distance r,

$$u^\mu(r) = \left(\left(1 - \frac{2M}{r}\right)^{-1/2},\ 0,\ 0,\ 0\right) = \left(1 - \frac{2M}{r}\right)^{-1/2} \xi^\mu.$$

The energy of the photon measured at r by stationary observer is

$$\hbar\omega(r) = -p \cdot u = -\left(1 - \frac{2M}{r}\right)^{-1/2} (\xi \cdot p)_r$$

> But $\xi \cdot p$ *is conserved along the photon geodesic* (ξ is a Killing vector) so $\xi \cdot p$ is in fact *independent of* r.

Therefore,

$$\hbar\omega_0 \equiv \hbar\omega(R_1) = -\left(1 - \frac{2M}{R_1}\right)^{-1/2}(\xi \cdot p)$$

$$\hbar\omega_\infty \equiv \hbar\omega(r \to \infty) = -(\xi \cdot p),$$

and from $\hbar\omega_\infty/\hbar\omega_0$ we obtain immediately a gravitational redshift

$$\omega_\infty = \omega_0 \left(1 - \frac{2M}{R_1}\right)^{1/2}.$$

> We have made no weak-field assumptions so this result should be valid for weak and strong fields.

For weak fields $2M/R_1$ is small, the square root can be expanded, and the G and c factors restored to give

$$\omega_\infty \simeq \omega_0 \left(1 - \frac{GM}{R_1 c^2}\right) \qquad \text{(valid for weak fields)}$$

which is the result derived earlier using the equivalence principle.

> By viewing ω as defining clock ticks, the redshift may also be interpreted as a *gravitational time dilation*.

9.1.4 Particle Orbits in the Schwarzschild Metric

Symmetries of the Schwarzschild metric:

1. Time independence $\longrightarrow$ Killing vector $\xi_t = (1,0,0,0)$
2. No dependence on $\varphi \longrightarrow$ Killing vector $\xi_\varphi = (0,0,0,1)$
3. Additional Killing vectors associated with full rotational symmetry (won't need in following).

Conserved quantities associated with these Killing vectors:

$$\varepsilon \equiv -\xi_t \cdot u = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

$$\ell \equiv \xi_\varphi \cdot u = r^2 \sin^2\theta \frac{d\varphi}{d\tau}.$$

Physical interpretation:

- At *low velocities* $\ell \sim$ (orbital angular momentum / unit rest mass)
- Since $E = p^0 = mu^0 = m dt/d\tau$,

$$\operatorname*{Lim}_{r\to\infty} \varepsilon = \frac{dt}{d\tau} = \frac{E}{m}$$

  and $\varepsilon \sim$ energy / (unit rest mass) *at large distance*.

Also we have the velocity normalization constraint

$$u \cdot u = g_{\mu\nu} u^\mu u^\nu = -1.$$

Conservation of angular momentum confines the particle motion to a plane, which we conveniently take to be the equatorial plane with $\theta = \frac{\pi}{2}$ implying that $u^2 \equiv u^\theta = 0$.

Then writing the velocity constraint

$$g_{\mu\nu}u^\mu u^\nu = -1.$$

out in the metric gives

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2(u^3)^2 = -1.$$

which we may rewrite using

$$u^\mu = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right)$$

$$\varepsilon = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \ell = r^2 \sin^2\theta \frac{d\varphi}{d\tau}.$$

in the form

$$\frac{\varepsilon^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(\frac{\ell^2}{r^2}+1\right)-1\right].$$

We can put this in the form

$$E = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r),$$

where we define a fictitious "energy"

$$E \equiv \frac{\varepsilon^2 - 1}{2}$$

and an effective potential

$$\begin{aligned} V_{\text{eff}}(r) &= \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(\frac{\ell^2}{r^2} + 1\right) - 1\right] \\ &= \underbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}}_{\text{Newtonian}} - \underbrace{\frac{M\ell^2}{r^3}}_{\text{correction}} \end{aligned}$$

> This is analogous to the energy integral of Newtonian mechanics with an effective potential V_{eff} and a proper time interval $d\tau$.

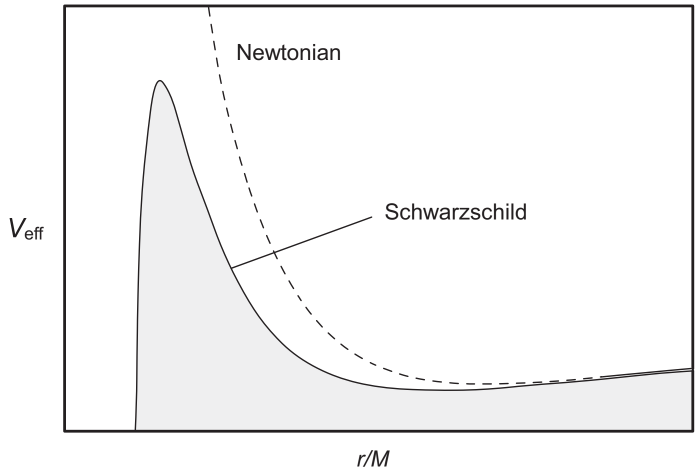

Figure 9.5: Effective potentials for finite ℓ in the Schwarzschild geometry and in Newtonian approximation.

Figure 9.5 compares the Schwarzschild effective potential with an effective Newtonian potential.

The Schwarzschild potential generally has one maximum and one minimum if $\ell/M > \sqrt{12}$.

Note the very different behavior of Schwarzschild and Newtonian mechanics at the origin because of the correction term in

$$V_{\text{eff}}(r) = \underbrace{-\frac{M}{r} + \frac{\ell^2}{2r^2}}_{\text{Newtonian}} - \underbrace{\frac{M\ell^2}{r^3}}_{\text{correction}}$$

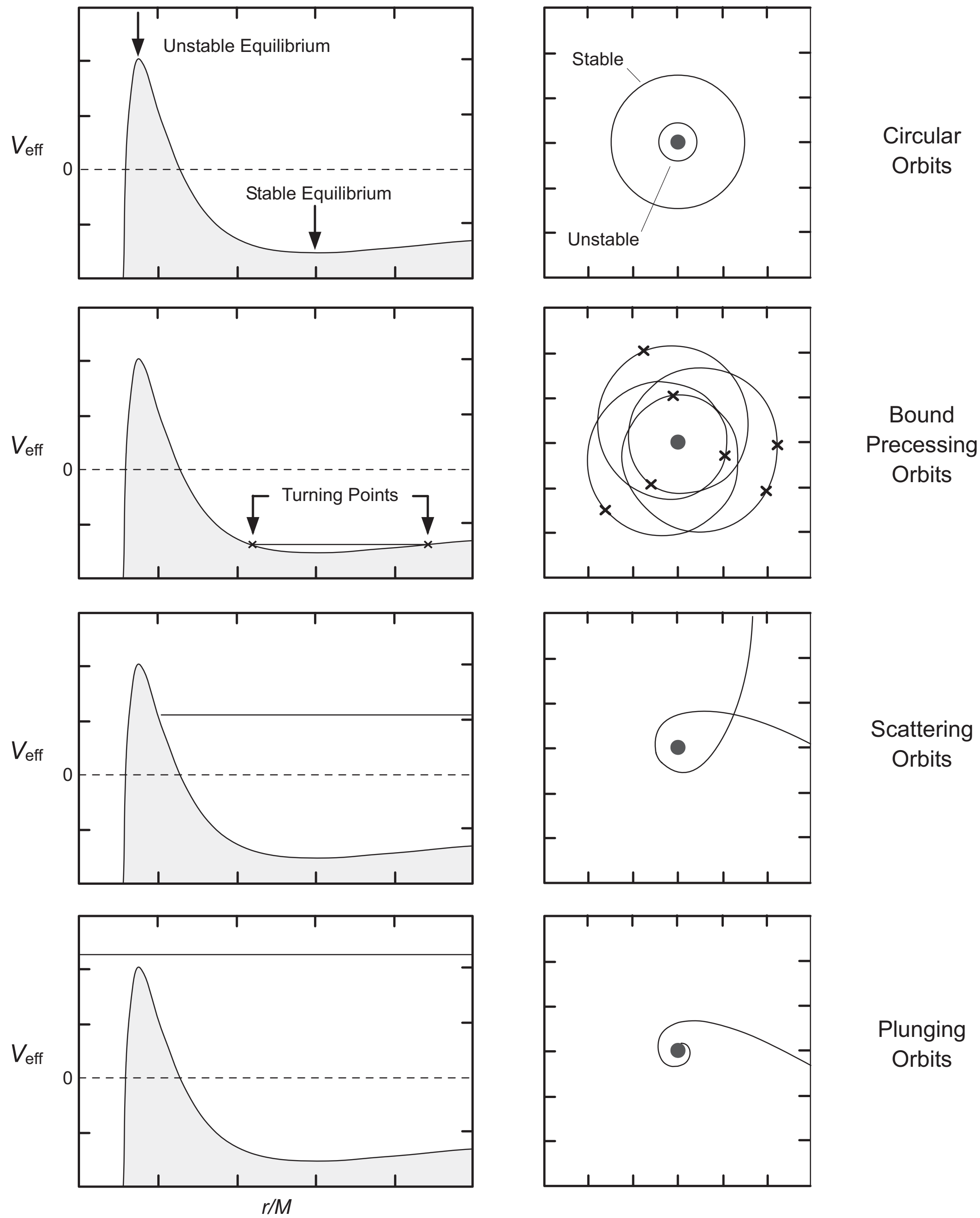

Figure 9.6: Orbits in a Schwarzschild spacetime. Effective potential on left and corresponding classes of orbits on right.

9.1.5 Innermost Stable Circular Orbit

The radial coordinate of the inner turning point for bound precessing orbits in the Schwarzschild metric is given by (Exercise)

$$r_- = \frac{\ell^2}{2M}\left(1 + \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right)$$

Thus r_- has a minimum possible value when

$$\frac{M}{\ell} = \frac{1}{\sqrt{12}}.$$

The corresponding radius for the *innermost stable circular orbit* $R_{\rm ISCO}$ is then

$$R_{\rm ISCO} = 6M.$$

The innermost stable circular orbit is important in determining how much gravitational energy can be extracted from matter accreting onto neutron stars and black holes.

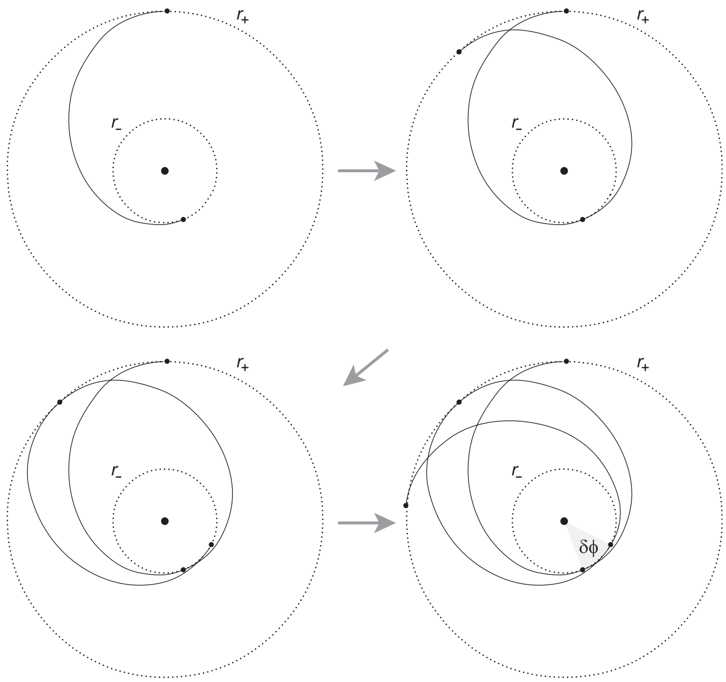

Figure 9.7: Precessing orbits in a Schwarzschild metric

9.1.6 Precession of Orbits

> An orbit closes if the angle φ sweeps out exactly 2π in the passage between two successive inner or two successive outer radial turning points.

> In Newtonian gravity the central potential is $1/r \rightarrow$ closed elliptical orbits. In Schwarzschild metric the effective potential deviates from $1/r$ and orbits *precess:* φ changes by more than 2π between successive radial turning points.

To investigate this precession quantitatively we require an expression for $d\varphi/dr$. From the energy equation

$$\frac{dr}{d\tau} = \pm\sqrt{2(E - V_{\text{eff}}(r))},$$

and from the conservation equation for ℓ,

$$\frac{d\varphi}{d\tau} = \frac{\ell}{r^2 \sin^2\theta}.$$

Combinine, recalling that we are choosing an orbital plane $\theta = \frac{\pi}{2}$,

$$\begin{aligned}
\frac{d\varphi}{dr} = \frac{d\varphi/d\tau}{dr/d\tau} &= \pm\frac{\ell}{r^2\sqrt{2(E - V_{\text{eff}}(r))}} \\
&= \pm\frac{\ell}{r^2}\left[2E - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) + 1\right]^{-1/2} \\
&= \pm\frac{\ell}{r^2}\left[\varepsilon^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\right]^{-1/2},
\end{aligned}$$

where we have used $E = \frac{1}{2}(\varepsilon^2 - 1)$.

The change in φ per orbit, $\Delta\varphi$, can be obtained by integrating over one orbit,

$$\begin{aligned}
\Delta\varphi &= \int_{r_-}^{r_+} \frac{d\varphi}{dr}\,dr + \int_{r_+}^{r_-} \frac{d\varphi}{dr}\,dr = 2\int_{r_-}^{r_+} \frac{d\varphi}{dr}\,dr \\
&= 2\ell \int_{r_-}^{r_+} \frac{dr}{r^2}\left[\varepsilon^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2} \\
&= 2\ell \int_{r_-}^{r_+} \frac{dr}{r^2}\left(\underbrace{c^2(\varepsilon^2-1) + \frac{2GM}{r} - \frac{\ell^2}{r^2}}_{\text{Newtonian}} + \underbrace{\frac{2GM\ell^2}{c^2r^3}}_{\text{correction}}\right)^{-1/2}
\end{aligned}$$

where in the last step G and c have been reinserted through the substitutions

$$M \longrightarrow \frac{GM}{c^2} \qquad \ell \longrightarrow \frac{\ell}{c},$$

Evaluation of the integral requires some care because *the integrand tends to* ∞ *at the integration limits:* From one of our earlier expressions

$$\frac{dr}{d\tau} = \pm\left[\varepsilon^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right)\right]^{1/2},$$

which is the denominator of our integrand. But the limits are turning points of the radial motion and $dr/d\tau = 0$ at r_+ or r_-.

In the Solar System and most other applications the values of $\Delta\varphi$ are very small, so it is sufficient to keep only terms of order $1/c^2$ beyond the Newtonian approximation.

Expanding the integrand and evaluating the integral with due care (Exercise) yields

$$\begin{aligned}\text{Precession angle} = \delta\varphi &\equiv \Delta\varphi - 2\pi \\ &\simeq 6\pi \left(\frac{GM}{c\ell}\right)^2 \ \text{rad/orbit.}\end{aligned}$$

This may be expressed in more familiar classical orbital parameters:

- In Newtonian mechanics $L = mr^2\omega$, where L is the angular momentum and ω the angular frequency.
- For Kepler orbits

$$\ell^2 = \left(\frac{L}{m}\right)^2 = \left(r^2\frac{d\varphi}{d\tau}\right)^2 \simeq \left(r^2\frac{d\varphi}{dt}\right)^2 = GMa(1-e^2),$$

  where e is the eccentricity and a is the semimajor axis.

This permits us to write

$$\begin{aligned}\delta\varphi &= \frac{6\pi GM}{ac^2(1-e^2)} \\ &= 1.861\times 10^{-7}\left(\frac{M}{M_\odot}\right)\left(\frac{\text{AU}}{a}\right)\frac{1}{1-e^2} \ \text{rad/orbit,}\end{aligned}$$

The form of

$$\delta\varphi = \frac{6\pi GM}{ac^2(1-e^2)}$$

shows explicitly that the amount of relativistic precession is enhanced by

- large M for the central mass,
- tight orbits (small values of a),
- large eccentricities e.

The precession observed for most objects is small.

- Precession of Mercury's orbit in the Sun's gravitational field because of general relativistic effects is observed to be *43 arcseconds per century.*
- The orbit of the Binary Pulsar precesses by about *4.2 degrees per year.*

The precise agreement of both of these observations with the predictions of general relativity is a strong test of the theory.

9.1.7 Escape Velocity in the Schwarzschild Metric

Consider a stationary observer at a Schwarzschild radial coordinate R who launches a projectile radially with a velocity v such that the projectile reaches infinity with zero velocity. This defines the escape velocity in the Schwarzschild metric.

- The projectile follows a radial geodesic since there are no forces acting on it
- The energy per unit rest mass is ε and it is conserved (time invariance of metric).
- At infinity $\varepsilon = 1$, since then the particle is at rest and the entire energy is rest mass energy. Thus $\varepsilon = 1$ at all times.

If u_{obs} is the 4-velocity of the stationary observer, the energy measured by the observer is

$$\begin{aligned} E &= -p \cdot u_{\text{obs}} = -mu \cdot u_{\text{obs}} \\ &= -mg_{\mu\nu} u^\mu u^\nu_{\text{obs}} \\ &= -mg_{00} u^0 u^0_{\text{obs}}, \end{aligned}$$

where $p = mu$, with p the 4-momentum and m the rest mass, and the last step follows because the observer is stationary. But

$$\underbrace{g_{00} = -\left(1 - \tfrac{2M}{r}\right)}_{\text{From metric}} \quad \underbrace{u^0_{\text{obs}} = \left(1 - \tfrac{2M}{R}\right)^{-1/2}}_{\text{Stationary observer}} \quad \underbrace{u^0 = \left(1 - \tfrac{2M}{r}\right)^{-1}}_{\text{From } \varepsilon = (1 - \frac{2M}{r})u^0 = 1}$$

Therefore,

$$
\begin{aligned}
E &= -mg_{00}u^0u^0_{\text{obs}} \\
&= m\left(1-\tfrac{2M}{r}\right)\left(1-\tfrac{2M}{r}\right)^{-1}\left(1-\tfrac{2M}{r}\right)^{-1/2} \\
&= m\left(1-\tfrac{2M}{R}\right)^{-1/2}
\end{aligned}
$$

But in the observer's rest frame

$$E = m\gamma = m(1-v^2)^{-1/2}$$

so comparison yields $2M/R = v^2$ and thus

$$v_{\text{esc}} = \sqrt{\frac{2M}{R}}$$

Notice that

- This, coincidentally, is the same result as for Newtonian theory.
- At the Schwarzschild radius $R = r_S = 2M$, the escape velocity is equal to c.

This is the first hint of an *event horizon* in the Schwarzschild spacetime.

9.1.8 Radial Fall of a Test Particle in Schwarzschild Geometry

It will be instructive for later discussion to consider the particular case of a radial plunge orbit that starts from infinity with zero kinetic energy ($\varepsilon = 1$) and zero angular momentum ($\ell = 0$).

First, let us find an expression for the proper time as a function of the coordinate r. From earlier expressions

$$E = \frac{\varepsilon^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3},$$

which implies for $\ell = 0$ and $\varepsilon = 1$,

$$\frac{dr}{d\tau} = \pm\left(\frac{2M}{r}\right)^{1/2}.$$

Choosing the negative sign (infalling orbit) and integrating with initial condition $\tau(r = 0) = 0$ gives

$$\frac{\tau}{2M} = -\frac{2}{3}\frac{r^{3/2}}{(2M)^{3/2}}$$

for the proper time τ to reach the origin as a function of the initial Schwarzschild coordinate r.

To find an expression for the coordinate time t as a function of r, we note that $\varepsilon = 1$ and is conserved. Then from

$$\varepsilon = 1 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \qquad \frac{dr}{d\tau} = \pm\left(\frac{2M}{r}\right)^{1/2}$$

we have that

$$\frac{dt}{dr} = \frac{dt/d\tau}{dr/d\tau} = -\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{-1/2},$$

which may be integrated to give

$$t = 2M\left(-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} + 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{(r/2M)^{1/2}+1}{(r/2M)^{1/2}-1}\right|\right).$$

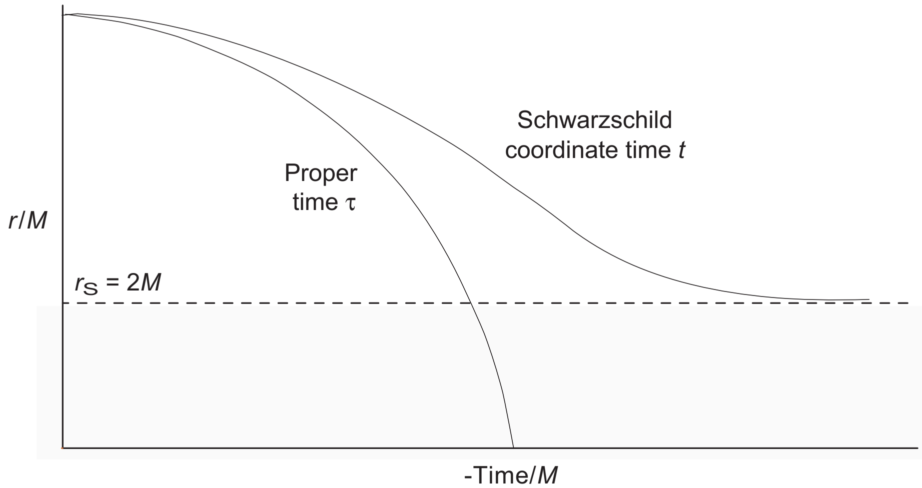

Figure 9.8: Comparison of proper time and Schwarzschild coordinate time for a particle falling radially in the Schwarzschild geometry.

- The proper time τ to fall to the origin is finite.
- For the same trajectory an infinite amount of coordinate time t elapses to reach the Schwarzschild radius.
- The smooth trajectory of the proper time through r_S suggests that the apparent singularity of the metric there is not real.

Later we shall introduce alternative coordinates that explicitly remove the singularity at $r = 2M$ (but not at $r = 0$).

9.1.9 Light Ray Orbits

Calculation of light ray orbits in the Schwarzschild metric largely parallels that of particle orbits, except that

$$u \cdot u = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0,$$

(not 1!) where λ is an affine parameter.

For motion in the equatorial plane $(\theta = \frac{\pi}{2})$, this becomes explicitly

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\varphi}{d\lambda}\right)^2 = 0.$$

By analogy with the corresponding arguments for particle motion

$$\varepsilon \equiv -\xi_t \cdot u = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda},$$

$$\ell = \xi_\varphi \cdot u = r^2 \sin^2\theta \frac{d\varphi}{d\lambda},$$

are conserved along the orbits of light rays.

With a proper choice of the normalization of the affine parameter λ, the conserved quantitity ε may be interpreted as the photon energy and the conserved quantity ℓ as its angular momentum at infinity.

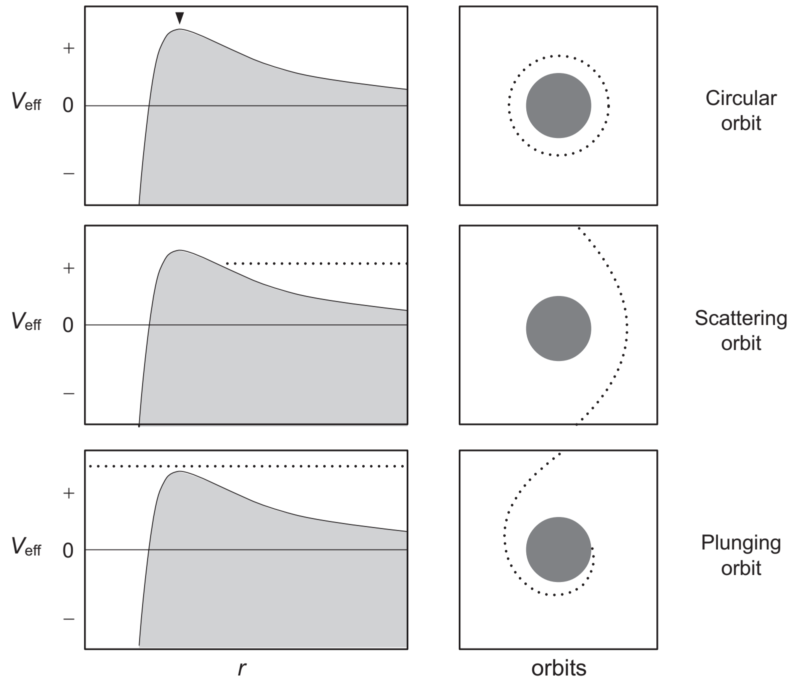

Figure 9.9: The effective potential for photons and some light ray orbits in a Schwarzschild metric. The dotted lines on the left side give the value of $1/b^2$ for each orbit.

By following steps analogous to the derivation of particle orbits

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}}(r)$$

$$V_{\text{eff}}(r) \equiv \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) \qquad b^2 \equiv \frac{\ell^2}{\varepsilon^2}.$$

The effective potential for photons and some classes of light ray orbits in the Schwarzschild geometry are illustrated in Fig. 9.9.

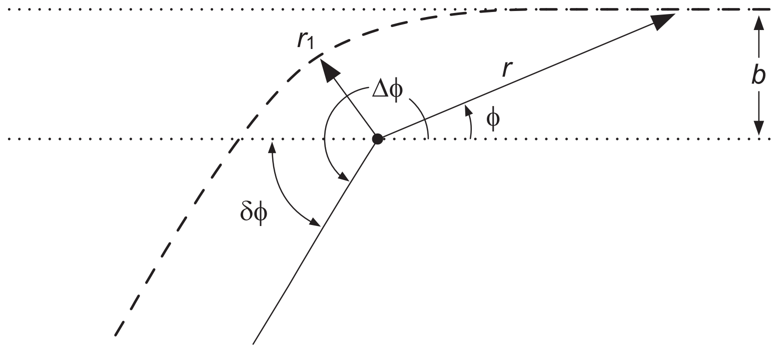

Figure 9.10: Deflection of light by an angle $\delta\varphi$ in a Schwarzschild metric.

9.1.10 Deflection of Light in a Gravitational Field

Proceeding in a manner similar to that for the calculation of the precession angle for orbits of massive objects, we may calculate the deflection $d\varphi/dr$ for a light ray in the Schwarzschild metric.

$$\delta\varphi = \frac{4GM}{c^2 b} = 2.970 \times 10^{-28} \left(\frac{M}{\mathrm{g}}\right)\left(\frac{\mathrm{cm}}{b}\right) \text{ radians}$$

$$= 8.488 \times 10^{-6} \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{b}\right) \text{ radians.}$$

For a photon grazing the surface of the Sun, $M = 1M_\odot$ and $b = 1R_\odot$, which gives $\delta\varphi \simeq 1.75$ arcseconds.

Observation of this deflection during a total solar eclipse catapulted Einstein to worldwide fame almost overnight in the early 1920s.