21 Lecture 21: The Schwarzschild Metric and Black Holes

"All of physics is either impossible or trivial. It is impossible until you understand it, and then it becomes trivial."

Ernest Rutherford

The Big Picture: Today we are starting the third (and last) part of the course: black holes, stars and galaxies. We show that the Einstein's field equations imply the existence of a space-time singularity, which we now know as "black holes".

The Schwarzschild Problem

Shortly after Einstein published his field equations of GR, Karl Schwarzschild solved them to find the space-time geometry outside a stationary, spherical distribution of matter of mass M. Since the space outside the distribution is empty, the energy-momentum tensor $T_{\alpha\beta}$ vanishes, so the Einstein's field equation becomes:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathcal{R} = 0, \tag{390}$$

with an appropriate metric tensor. The appropriate boundary conditions are:

1. metric must match interior metric at the body's surface;
2. metric must go to the flat (Minkowski) metric far away from the body.

We now solve for the Schwarzschild metric $g_{\alpha\beta}$ which solves the Schwarzschild problem. We start with a general static and isotropic metric:

1. **static:** *both* time-independent *and* symmetric under time reversal (*only* time-independent $\Longleftrightarrow$ *stationary*);
2. **isotropic:** invariant under spatial rotations (same in all directions).

The interval satisfying these criteria may be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{391}$$

where the first two term on the RHS describe radial behavior (isotropy), and the last two the surface of the sphere (spherical symmetry). It can be expressed in many equivalent forms. One convenient form is:

$$ds^2 = -e^{N(r)}dt^2 + e^{P(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{392}$$

corresponding to the metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} -e^{N(r)} & 0 & 0 & 0 \\ 0 & e^{P(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{393}$$

The Schwarzschild problem reduces to solving for $N(r)$ and $P(r)$ from Einstein's field equations and the appropriate boundary conditions.

Solving the Schwarzschild Problem

Earlier we have defined an alternative Lagrangian [eq. 26]:

$$L = \frac{1}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta, \tag{394}$$

(where dot denotes s-derivative) which for the metric in eq. (393) becomes ($x^0 \to t$, $x^1 \to r$, $x^2 \to \theta$, $x^3 \to \phi$):

$$L = -\frac{1}{2} e^N \dot{t}^2 + \frac{1}{2} e^P \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{1}{2} r^2 \sin^2\theta \dot{\phi}^2, \tag{395}$$

This alternative Lagrangian allows us to easily read off Christoffel symbols by comparing it to the geodesic equation [eq. (31)]:

$$\frac{d^2 x^\nu}{ds^2} + \Gamma^\nu_{\gamma\delta} \frac{dx^\gamma}{ds} \frac{dx^\delta}{ds} = 0, \tag{396}$$

which we can combine to obtain the Riemann and Ricci tensors. Let us solve the Lagrange equations

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^\alpha} = 0,$$

for each of the components of the space-time ($'$ denotes r-derivative):

- t-component:

$$\begin{aligned} \frac{\partial L}{\partial t} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right) &= 0 \\ 0 - \frac{d}{ds}\left(-e^N \dot{t}\right) &= 0 \\ e^N \frac{dN}{dr} \frac{dr}{ds} \dot{t} + e^N \ddot{t} &= 0 \\ e^N \left(\ddot{t} + N' \dot{t} \dot{r}\right) &= 0 \\ \Longrightarrow \quad \frac{d^2 t}{ds^2} + N' \left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right) &= 0. \end{aligned} \tag{397}$$

After comparing it to eq. (396), we obtain

$$\frac{d^2 t}{ds^2} + \left(\Gamma^0_{01} + \Gamma^0_{10}\right) \left(\frac{dt}{ds}\right)\left(\frac{dr}{ds}\right) = 0, \tag{398}$$

which means that (because of symmetry of the Christoffel symbols: $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$)

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} N', \tag{399}$$

while other $\Gamma^0_{\alpha\beta}$ symbols vanish.

- r-component:

$$\begin{aligned}
\frac{\partial L}{\partial r} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{r}}\right) &= 0 \\
-\frac{1}{2}N'e^N\dot{t}^2 + \frac{1}{2}P'e^P\dot{r}^2 + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 - \frac{d}{ds}\left(e^P\dot{r}\right) &= 0 \\
-\frac{1}{2}N'e^N\dot{t}^2 + \frac{1}{2}P'e^P\dot{r}^2 + r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 - e^PP'\dot{r}^2 - e^P\ddot{r} &= 0 \\
-e^P\left(\ddot{r} + \frac{1}{2}N'e^{N-P}\dot{t}^2 + \frac{1}{2}P'\dot{r}^2 - e^{-P}r\dot{\theta}^2 - e^{-P}r\sin^2\theta\dot{\phi}^2\right) &= 0 \\
\frac{d^2r}{ds^2} + \frac{1}{2}N'e^{N-P}\left(\frac{dt}{ds}\right)^2 + \frac{1}{2}P'\left(\frac{dr}{ds}\right)^2 - e^{-P}r\left(\frac{d\theta}{ds}\right)^2 - e^{-P}r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 &= 0. \qquad (400)
\end{aligned}$$

After comparing it to eq. (396), we obtain

$$\frac{d^2r}{ds^2} + \Gamma^1_{00}\left(\frac{dt}{ds}\right)^2 + \Gamma^1_{11}\left(\frac{dr}{ds}\right)^2 + \Gamma^1_{22}\left(\frac{d\theta}{ds}\right)^2 + \Gamma^1_{33}\left(\frac{d\phi}{ds}\right)^2 = 0, \qquad (401)$$

which means that

$$\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2}N'e^{N-P}, \\
\Gamma^1_{11} &= \frac{1}{2}P', \\
\Gamma^1_{22} &= -e^{-P}r, \\
\Gamma^1_{33} &= -e^{-P}r\sin^2\theta,
\end{aligned} \qquad (402)$$

while other $\Gamma^1_{\alpha\beta}$ symbols vanish.

- θ-component:

$$\begin{aligned}
\frac{\partial L}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 0 \\
\frac{1}{2}r^2 2\sin\theta\cos\theta\dot{\phi}^2 - \frac{d}{ds}\left(r^2\dot{\theta}\right) &= 0 \\
\frac{1}{2}r^2\sin 2\theta\dot{\phi}^2 - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} &= 0 \\
-r^2\left(\ddot{\theta} - \frac{1}{2}\sin 2\theta\dot{\phi}^2 + 2\frac{\dot{r}}{r}\dot{\theta}\right) &= 0 \\
\frac{d^2\theta}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) - \frac{1}{2}\sin 2\theta\left(\frac{d\phi}{ds}\right)^2 &= 0 \qquad (403)
\end{aligned}$$

After comparing it to eq. (396), we obtain

$$\frac{d^2\theta}{ds^2} + \left(\Gamma^2_{12} + \Gamma^2_{21}\right)\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) + \Gamma^2_{33}\left(\frac{d\phi}{ds}\right)^2 = 0, \qquad (404)$$

which means that

$$\begin{aligned}
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r}, \\
\Gamma^2_{33} &= -\frac{1}{2}\sin 2\theta,
\end{aligned} \qquad (405)$$

while other $\Gamma^2_{\alpha\beta}$ symbols vanish.

- ϕ-component:

$$\begin{aligned}
\frac{\partial L}{\partial \phi} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= 0 \\
0 - \frac{d}{ds}\left(r^2 \sin^2\theta \dot{\phi}\right) &= 0 \\
-2r\dot{r}\sin^2\theta\dot{\phi} - 2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} - r^2\sin^2\theta\ddot{\phi} &= 0 \\
-r^2\sin^2\theta\left(\ddot{\phi} + 2\frac{\dot{r}}{r}\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}\right) &= 0 \\
\frac{d^2\phi}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right) + 2\cot\theta\left(\frac{d\theta}{ds}\right)\left(\frac{d\phi}{ds}\right) &= 0 \qquad (406)
\end{aligned}$$

After comparing it to eq. (396), we obtain

$$\frac{d^2\phi}{ds^2} + \left(\Gamma^3_{13} + \Gamma^3_{31}\right)\left(\frac{dr}{ds}\right)\left(\frac{d\phi}{ds}\right) + \left(\Gamma^3_{23} + \Gamma^3_{32}\right)\left(\frac{d\theta}{ds}\right)\left(\frac{d\phi}{ds}\right) = 0, \qquad (407)$$

which means that

$$\begin{aligned}
\Gamma^3_{13} &= \Gamma^3_{31} = \frac{1}{r}, \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta,
\end{aligned} \qquad (408)$$

while other $\Gamma^3_{\alpha\beta}$ symbols vanish.

These Christoffel symbols associated with the metric given in eq. (393) are needed to compute the Riemann tensor, which, in turn, is used to compute the Ricci tensor and Ricci scalar, to fully determine the LHS of the Einstein's equation: $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R} = 0$.

It can be shown (Homework set #3) that $G_{\alpha\beta} = 0$ leads to

$$\begin{aligned}
-\frac{e^{N-P}}{r}\left(P' - \frac{1}{r}\right) - -\frac{e^N}{r^2} &= 0, \\
-\frac{N'}{r} - \frac{1}{r^2}\left(1 - e^P\right) &= 0, \\
-\frac{1}{2}r^2 e^{-P}\left[N'' - \frac{1}{2}P'N' + \frac{1}{2}\left(N'\right)^2 + \frac{N' - P'}{r}\right] &= 0. \qquad (409)
\end{aligned}$$

These expressions combine to give (Homework set #3) to obtain

$$\frac{dP}{dr} = -\frac{dN}{dr} = \frac{1}{r}\left(1 - e^P\right),$$

which can be solved for P:

$$\begin{aligned}
\int \frac{dP}{1 - e^P} &= \int \frac{dr}{r} \\
\int \left(\frac{1 - e^P}{1 - e^P} + \frac{e^P}{1 - e^P}\right) dP &= \ln Cr \\
P - \ln\left(1 - e^P\right) = \ln e^P - \ln\left(1 - e^P\right) = \ln\frac{e^P}{1 - e^P} &= \ln Cr \\
\frac{e^P}{1 - e^P} = Cr \quad \Longrightarrow \quad e^P = \frac{Cr}{1 + Cr} & \qquad (410)
\end{aligned}$$

Solving for N we obtain

$$N = -P + const. \Longrightarrow e^N = e^{const} e^{-P}, \tag{411}$$

but since we have to recover Minkowski metric at large distances:

$$\begin{aligned} \lim_{r\to\infty} g_{00} &\to -1, \\ \lim_{r\to\infty} g_{11} &\to 1, \end{aligned} \tag{412}$$

and $const. = 0$. Therefore,

$$\begin{aligned} N &= -P \\ g_{00} &= -e^N = -e^{-P} = -\frac{1}{g_{11}} = -\left(\frac{1+Cr}{Cr}\right) = -\left(1+\frac{1}{Cr}\right). \end{aligned} \tag{413}$$

For weak gravitational fields, we derived in eq. (39) including the constants:

$$g_{00} = -\left(1+\frac{2\Phi}{c^2}\right) = -\left(1-\frac{2GM}{rc^2}\right) = -\frac{1}{g_{11}} \qquad \Longrightarrow \qquad C = -\frac{c^2}{2GM}. \tag{414}$$

We finally arrive at the solution to the Schwarzschild problem, and the corresponding line element in the Schwarzschild metric (with constants c and G included explicitly):

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2dt^2 + \frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{415}$$

Birkhoff's theorem. The derivation of the Schwarzschild metric does not require any other information about the distribution of the matter giving rise to the gravitational field — it only requires that it is:

- spherically symmetric;
- that it has zero density at the radius of interest.

Birkhoff showed that *any* spherically symmetric *vacuum* solution of Einstein's field equations *must also be static* and agree with Schwarzschild's solution. Therefore, the spherically symmetric mass leads to the Schwarzschild metric *regardless* of whether the mass is static, collapsing, expanding or pulsating. This, of course, refers to the field *outside* the mass, as first stated in the derivation, because we start with $T_{\alpha\beta} = 0$. Two of the most important features of Newtonian gravity therefore apply to GR:

- the gravity of a spherical body appears to act from a central point mass;
- the gravitational field inside a spherical shell vanishes.

Schwarzschild Radius, Event Horizon and Black Holes

The Schwarzschild space-time metric has a singularity when the denominator in the second term is equal to zero:

$$1 - \frac{2GM}{c^2r} = 0, \tag{416}$$

which happens when the radius associated with mass M is

$$r_s = \frac{2GM}{c^2}. \tag{417}$$

This is called the *Schwarzschild radius*, or the *event horizon*, because events occurring inside it cannot propagate light signals to the outside. Any body which is small enough to exist within its own event horizon is therefore disconnected from the rest of the Universe: its only physical manifestation is through its (infinitely) deep gravitational potential well, which is what led to the adoption of the term *black hole* in the late 1960's.

For a body with mass equal to that of our Sun, the event horizon is equal to

$$r_s = \frac{2GM_\odot}{c^2} = \frac{2\left(6.67\times 10^{-8}\right)\left(2\times 10^{33}\right)}{\left(3\times 10^{10}\right)^2} \approx 3\times 10^5 \text{cm} = 3 \text{ km}. \tag{418}$$

We can write the proper time in the Schwarzschild metric as

$$ds^2 = -d\tau^2 \qquad \Longrightarrow \qquad d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \frac{dr^2}{1-\frac{2GM}{c^2r}} - -r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{419}$$

where dt is the time interval according to an observer at $r \to \infty$, and $d\tau$ is the time interval measured by a *local* observer (in comoving coordinates, in which the Universe is static). Because for the local observer the Universe is static, it means that $dr = 0$, so

$$dt^2 = \frac{d\tau^2}{1-\frac{2GM}{c^2r}}. \tag{420}$$

This is *time dilation*: while the local observer near the black hole (at $r \gtrsim r_s$) sees nothing unusual about her/his time-measurements ($d\tau$), the measurements of the observer at $r \to \infty$ would suggest that the local observer's clock runs slow by a factor $\left(1 - \frac{2GM}{c^2r}\right)^{-1/2}$. It becomes *infinitely* slow at the event horizon r_s. Therefore, the inertial observer (at infinity) can *never* witness the infalling observer reach the event horizon.

Orbits in Schwarzschild's Geometry

For the dynamics of black holes and their accretion disks, it is important to quantify the motion of particles which find themselves near the black hole. We now present a brief exposition of the orbit theory near a black hole.

In order to compute orbits in Schwarzschild's geometry, we need to first compute the equations of motion.

Combining components of the solutions to Einstein's equation in Schwarzschild's metric which we just derived with the general property of massive particles in a metric

$$g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1, \tag{421}$$

(= 0 for photons), it can be shown that the motion near the black hole can be described with

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \bar{B}^2 - \left(1 - \frac{\bar{A}^2}{r^2}\right)\left(1 - \frac{r_s}{r}\right) \\ \frac{d\phi}{d\tau} &= \frac{\bar{A}}{r^2}, \end{aligned} \tag{422}$$

where $\bar{A}$ is the angular momentum per unit mass and $\bar{B}^2$ is the energy per unit mass relative to infinity.

We now define a *relativistic potential*

$$V(r) \equiv \left(1 + \frac{\bar{A}^2}{r^2}\right)\left(1 - \frac{r_s}{r}\right) \tag{423}$$

so that

$$\left(\frac{dr}{d\tau}\right)^2 = \bar{B}^2 - V(r). \tag{424}$$

The shape of the potential is given in Fig. 39. The two minima of the potential $(r/r_s)_\pm$ are found

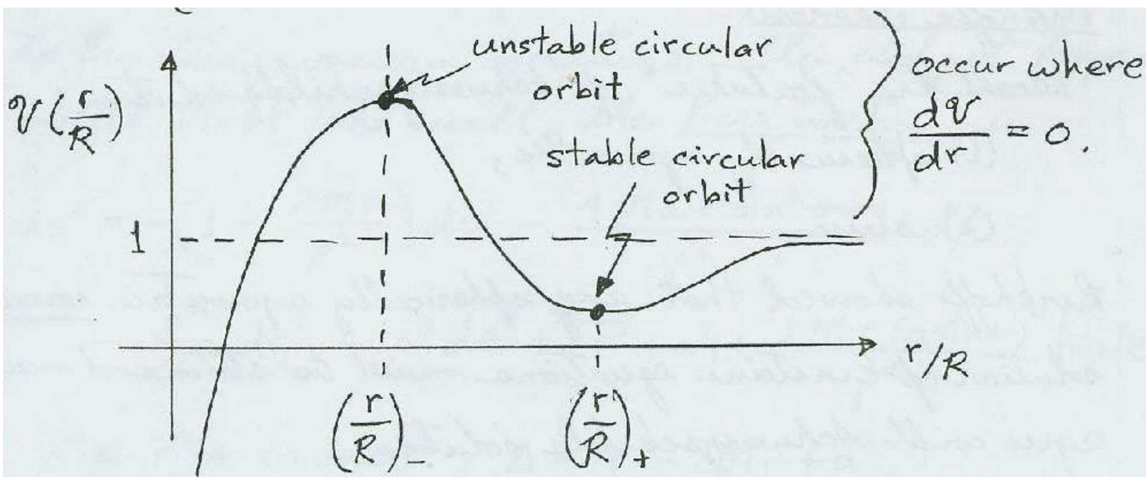

Figure 39: Relativistic potential $V(r)$.

by solving:

$$\begin{aligned} \frac{dV}{dr} &= \frac{r_s}{r^2} - \frac{2\bar{A}^2}{r^3} + \frac{3r_s\bar{A}^2}{r^4} \\ &\Longrightarrow \left(\frac{r}{r_s}\right)^2 - 2\left(\frac{\bar{A}}{r_s}\right)^2\left(\frac{r}{r_s}\right) + 3\left(\frac{\bar{A}}{r_s}\right)^2 = 0. \\ &\Longrightarrow \left(\frac{r}{r_s}\right)_\pm = \frac{\bar{A}}{r_s}\left[1 \pm \sqrt{1 - \frac{3}{\frac{\bar{A}}{r_s}},}\right] \end{aligned} \tag{425}$$

so there are *no* circular orbits if $\frac{\bar{A}}{r_s} < \sqrt{3}$.