A brief discussion on Gauge Theories

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This is a brief discussion on Gauge Theories. It aims to complement course material and provide a short insight into what gauge theories are, how the work and why we use them. The technical details are described in Prof. Osborn's AQFT notes, which is reference [1].

1 What's the big deal with gauge theories?

First, we will briefly discuss some points about gauge theories and then derive the form that gauge theories must take. The discussion will probably be easier to understand once you read Section (2.2) onwards.

So, what's the big deal with gauge theories all about? Well, gauge theories appear to describe three of the four fundamental forces of nature pretty well. Those three forces are electromagnetism, the weak force and the strong force. These are described theoretically by the standard model. The standard model, particularly QED, is verified to a high precision, within current experimental capabilities that is.

Historically, while trying to explain the quatum effects of electrodynamics, it was found QED could be explained by a $U(1)$ abelian gauge theory. Yang and Mills then generalised this abelain $U(1)$ gauge theory to the non abelian gauge theory case.

In the section on gauge theories in [1], it is shown that gauge transformations leave the action and the classical equations of motion invariant. Thus the dynamical variables are not the gauge fields, but rather the gauge fields unrelated by a gauge transformation. The gauge fields unrelated by a gauge transformation are called $\mathcal{A}/\mathcal{G}$ in [1]. We can remove these redunant degrees of freedom by fixing a gauge. The gauge is fixed by using the Feddeev-Popov method, as described in [1]. Since the action is invariant under a gauge transformation, we can choose to fix any gauge we want. The choice of gauge is arbitrary and is chosen to make the algebra easier. We have a price to pay for fixing the gauge. We introduce unphysical "ghost" fields into our theory. Ghost fields decouple in the abelian $U(1)_{EM}$ gauge theory that describes QED[1]. After gauge fixing, there is a 'left-over' gauge symmetry called BRST symmetry. This symmetry can be used to show that the physical Hilbert space consisting only of positive norm states are void of ghosts.

One of the major differences between abelian and non-ableian gauge theories is the existence of self interactions in the non-abelian case[2].

What is a "gauge theory"? A gauge theory is a theory where the action is invariant under a continuous group symmetry that depends on spacetime. This is described in more detail in Section(2.2). When the symmetry group depends on spacetime, it is called a local symmetry. The continuous symmetry that depends on spacetime is called a gauge group. The transformation that depends on spacetime is called a gauge transformation. You get the point!

New gauge fields $A^a_\mu(x)$ are introduced in order to make the theory invariant under gauge transformations. An interaction term of the form $g\bar{\Psi}(x)A^a_\mu(x)\Psi(x)$ is seen in the gauge theory. These gauge fields mediate a force between the fields $\bar{\Psi}$, Ψ with coupling g.

As described in Section (3), the gauge fields are massless, unless spontaneous symmetry breaking (SSB) occurs. If SSB occurs, then the gauge fields with a generator that respects the vacuum will remain massless. In contrast, a gauge field with a corresponding generator that does not respect the vacuum will gain a mass. For example, after SSB, the $W^\pm, Z^0$ are massive but the photon is massless in electroweak theory. [3]

- A **global** symmetry is a symmetry which does not depend on spacetime. Global symmetries give rise to conserved **currents** and **charges** as described by Noether's theorem.

- A gauge symmetry is one where the symmetry group is continuous and depends on spacetime, i.e a continuous local symmetry. Gauge symmetries introduce **gauge fields** to the theory which **mediate** a force.

[1] The ghost fields couple to gauge fields through the structure constants f^a_{bc}. These are zero in abelian case.
[2] See [1] for a description of ghosts and self interactions.
[3] For more on this, see page 31 in the solutions to the third example sheet for the SM course.

1.1 Examples of Gauge Theories

1.1.1 QED

QED has the gauge group $= U(1)_{EM}$. The number of gauge fields is $dim(U(1)_{EM}) = 1$. This gauge field is the photon. It couples to charged leptons and quarks. Does SSB occur: No. So the photon remains massless.

1.1.2 QCD

QCD has the gauge group $SU(3)_{colour}$. A gauge transformation is $U \in SU(3)_{colour}$. QCD offers a new way of thinking about matter. Every quark field of flavour f, say $\Psi^f(x)$, has an associated colour of **red**, **green** or **blue**. Define

$$\chi^f(x) = \begin{pmatrix} \Psi^f_{\mathbf{red}}(x) \\ \Psi^f_{\mathbf{green}}(x) \\ \Psi^f_{\mathbf{blue}}(x) \end{pmatrix}$$

Construct the gauge invariant lagrangian from $\chi^f(x)$. This is

$$\mathcal{L} = \overline{\chi}^f(i\gamma^\mu D_\mu - m)\chi^f$$

This Lagrangian is invariant under $SU(3)_{colour}$, i.e, $\chi^f(x) \to U(x)\chi^f(x)$ where $U(x) \in SU(3)_{colour}$. Number of gauge fields $= dim(SU(3)_{colour}) = 8$. These are the eight gluon fields that couple to the quarks, holding them together to form non-pertubative bound states called hadrons. Does SSB occur? No, so the gluons remain massless.

1.1.3 ElectroWeak (EW) Theory

The EW theory has the gauge group $SU(2)_{left} \times U(1)_{hypercharge}$. The number of gauge fields is equal to $dim(SU(2)_{left} \times U(1)_{hypercharge}) = 4$. The gauge fields are the W^a_μ, B_μ, $a = 1, 2, 3$.

Does SSB occur? Yes, the $SU(2)_{left} \times U(1)_{hypercharge}$ theory interacts with a complex doublet. A complex doublet has four real scalar field components. The symmetry group is spontaneously broken to $U(1)_{\text{QED}}$. Each broken generator produces a goldstone boson. The number of goldstone bosons is $dim(G/H) = dim(SU(2)_{left} \times U(1)_{hypercharge}) - dim(U(1)_{\text{QED}}) = 4 - 1 = 3$. These three goldstone bosons are "eaten" by three gauge fields to become massive (in unitary gauge). We still have one massive real scalar field left after SSB, which we call the Higgs.

After SSB, the massive gauge fields are called $W^\pm_\mu, Z^0_\mu$ while the massless gauge field is called the photon. The $W^\pm_\mu$ couple to left handed matter causing flavour changing processes like beta decay, the Z^0_μ couples to all particles and the photon couples to charged matter.

2 The derivition of Gauge Theories

Most of this section is nearly stolen fully from [2]. For a further and fuller discussion, read [2].

What excatly is a gauge theory? And how do gauge fields get introduced into our theory? How would you find a gauge theory? Starting our story on global symmetries will help clarify ideas before we progress to gauge symmetries.

2.1 Global symmetries

Let's start with a free fermion field described by the Dirac lagrangian

$$\mathcal{L} = \overline{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x)$$

A symmetry is a transformation of the fields that leaves the action (and hence "physics") invariant. A global symmetry is a symmetry that does not depend on spacetime. An example of a global symmetry is the phase transformation

$$\Psi(x) \to e^{i\alpha}\Psi(x) \quad ; \qquad \overline{\Psi}(x) \to e^{-i\alpha}\overline{\Psi}(x)$$

where α does not depend on spacetime, x. Then

$$\begin{aligned} \mathcal{L} &= \overline{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x) \\ &\to \overline{\Psi}(x)e^{-i\alpha}(i\gamma^\mu\partial_\mu - m)e^{i\alpha}\Psi(x) \\ &= \mathcal{L} \end{aligned}$$

Since α did not depend on spacetime, x, we could commute it past the derivitive and so this phase transformation was a symmetry of $\mathcal{L}$. Now, let's get a little more complicated. Not much though, we will take little steps. Say we have two fields $\Psi_1(x)$ and $\Psi_2(x)$ both with the same mass $m_1 = m_2 = m$. Define

$$\chi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} \quad \overline{\chi}(x) = \begin{pmatrix} \overline{\Psi}_1(x) & \overline{\Psi}_2(x) \end{pmatrix}$$

Then the new lagrangian for the two free fields is

$$\begin{aligned} \mathcal{L} &= \overline{\Psi}_1(x)(i\gamma^\mu\partial_\mu - m)\Psi_1(x) + \overline{\Psi}_2(x)(i\gamma^\mu\partial_\mu - m)\Psi_2(x) \\ &= \overline{\chi}(x)\left[(i\gamma^\mu\partial_\mu - m)\mathbb{1}_{2x2}\right]\chi(x) \end{aligned}$$

$\chi(x)$ is an 8 component column vector. γ^μ is a 4×4 matrix. We can also think of $\chi(x)$ as being a two component spinor-column vector, where each component is a 4-component column vector Ψ. Then we can think of γ^μ as a scalar with respect to the spinor doublet $\chi(x)$. When thinking of $\chi(x)$ as a two component spinor-vector, then $(i\gamma^\mu\partial_\mu - m)$ is a scalar and $(i\gamma^\mu\partial_\mu - m)\mathbb{1}_{2x2}$ is a scalar by the identity matrix. The $\mathbb{1}_{2x2}$ can be left out but we'll leave it in, to be explicit. In this case, we have a $U(2)$ global symmetry given by

$$\begin{aligned} \chi(x) &\to V\chi(x); \quad \text{where} V \in U(2) \\ \overline{\chi}(x) &\to \overline{\chi}(x)V^\dagger \quad \text{and so} \\ \mathcal{L} &\to \overline{\chi}(x)V^\dagger\left[(i\gamma^\mu\partial_\mu - m)\right]V\chi(x) = \mathcal{L} \end{aligned}$$

- We can think of $\chi(x)$ as being a two component spinor-column vector. Then we can think of γ^μ as a scalar with respect to χ. When thinking in this way, we can commute γ^μ past V. See page 6.3 in the solutions to example sheet one.
- Since V doesn't depend on spacetime, we can commute it past the derivitive resulting in $\mathcal{L}$ being invariant.
- $U(N) = SU(N) \times U(1)$ so we can factor out a phase from $U(2)$.
- This $U(2)$ symmetry can be extended to a $U(N)$ symmetry straightforwardly. Given N fermion fields with the same mass, put them into an N component column vector $\chi(x)$ and just redo everything we have done.
- This is $SU(3)$ isospin symmetry when we consider the u, d, s quarks as having equal mass.

But what if our symmetry transformation did depend on spacetime? Then we couldn't commute it past the derivitive so easily. What would happen to our symmetry?

2.2 Gauge symmetry

A local symmetry is a symmetry that depends on spacetime, x. A gauge symmetry is a local symmetry where the symmetry group is continous, e.g, $U(N), SU(N)$.

Let's say we have an N component column vector of fermion fields with identical masses:

$$\chi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{pmatrix}$$

Then, just as in the previous example, the Lagrangian

$$\mathcal{L} = \overline{\chi}(x)\left[(i\gamma^\mu\partial_\mu - m)\mathbb{1}_{NxN}\right]\chi(x)$$

will be invariant under a global U(N) transformation given by

$$\begin{aligned} \chi(x) &\to V\chi(x); \quad \text{where} V \in U(N) \\ \overline{\chi}(x) &\to \overline{\chi}(x)V^\dagger \end{aligned}$$

Promote the transformation V to be a local symmetry, i.e, it will depend on spacetime: $V \to V(x)$. How does the Lagrangian transform now?

$$\begin{aligned}\mathcal{L} &\to \overline{\chi}(x)V^{\dagger}(x)\left[i\gamma^{\mu}\partial_{\mu} - m\right]V(x)\chi(x) \\ &= \overline{\chi}(x)V^{\dagger}(x)\left[i\gamma^{\mu}\partial_{\mu}(V(x)\chi(x)) - mV(x)\chi(x)\right] \\ &= \overline{\chi}(x)V^{\dagger}(x)\left[i\gamma^{\mu}V(x)\partial_{\mu}\chi(x) + i\gamma^{\mu}\partial_{\mu}(V(x))\chi(x) - mV(x)\chi(x)\right] \\ &= \mathcal{L} + \overline{\chi}(x)V^{\dagger}(x)i\gamma^{\mu}\partial_{\mu}(V(x))\chi(x)\end{aligned}$$

which is not invariant!! The main issue is that we cannot commute the transformation past the derivitive because it depends on spacetime.

This seems like a dead end. The local symmetry isn't even a symmetry! But let's say we are really adamant that "physics" is invariant under

$$\chi(x) \to V(x)\chi(x)$$

We want to be able to transform each $\chi(x)$ at each spacetime point, x, seperately and for "physics" to be invariant. What can we do to enforce this? The problem lies with the derivitive. The derivitive does not transform **covariantly**. We need a derivitive that transforms covariantly in order to make our Lagrangian invariant under gauge transformations. If we can construct a covariant derivitive, call it D_{μ}, then $\mathcal{L}$ will remain invariant. What we have so far:

$$\begin{aligned}\partial_{\mu}(\chi(x)) &\not\to V(x)\partial_{\mu}\chi(x); \quad \partial_{\mu} \text{ does not transform covariantly} \\ D_{\mu}(\chi(x)) &\to V(x)D_{\mu}\chi(x); \quad \text{while } D_{\mu} \text{ does.}\end{aligned}$$

We need to construct a new derivitive anyway because the ordinary derivitive, ∂_{μ}, doesn't make sense anymore. For example, consider the ordinary derivitive in some direction n^{μ}:

$$n^{\mu}\partial\mu\chi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\chi(x+\epsilon n) - \chi(x)\right)$$

But since our new theory is invariant under local transformations, we can transform $\chi(x+\epsilon n)$ and $\chi(x)$ independently. The result is that the derivitive, ∂_{μ}, loses it's normal meaning. It doesn't make sense to compare two fields at different spacetime points anymore. But if we construct a new derivitive where the two fields transform in the same way, then the derivitive would make sense. Hopefully, this new derivitive that we construct will also transform covariantly!

Somehow, we must make $\chi(x)$ and $\chi(x+\epsilon n)$ transform the same way. Define $U(y,x) \in SU(N)$ (called a parallel transport) which transforms under a gauge transformation as

$$U(y,x) \to V(y)U(y,x)V(x)^{\dagger} \tag{1}$$

We say $U(y,x)$ 'transports'the gauge transformation from $x \to y$ in the following sense:

$$\begin{aligned}U(y,x)\chi(x) &\to V(y)U(y,x)V^{\dagger}(x)V(x)\chi(x) \\ &= V(y)U(y,x)\chi(x)\end{aligned}$$

So $U(y,x)\chi(x)$ transforms the same as $\chi(y)$. Now $\chi(y) - U(y,x)\chi(x)$ transforms like $\chi(y)$ and as such is well defined. But this is excatly what we need for the derivitive to work (what a coincedence!). Take $y = x + \epsilon n$. and define the new covariant derivitive as

$$n^{\mu}D_{\mu}\chi(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\chi(x+\epsilon n) - U(x+\epsilon n, x)\chi(x)\right] \tag{2}$$

We see that under a gauge transformation $D_{\mu}\chi(x) \to V(x)D_{\mu}\chi(x)$. This is by construction as we have defined each term to transform the same. So the covariant derivitive does transform covariantly! If we can find an explicit form for $U(y,x)$, then we could find an explicit form for D_{μ}. Then we would have a Lagrangian that would be invariant under local symmetries. It is found that[4]

$$U_{\Gamma}(y,x) = \exp(ig\int_{\Gamma} A_{\mu}(x)dx^{\mu})$$

where Γ is a path joining y to x. Since $U(y,x) \in SU(N)$, $A_{\mu}(x)$ is an element of the Lie algebra of $SU(N)$ with coupling g. Infinitisimal paths between x and y yield

$$\begin{aligned}U(x+\epsilon n, x) &\approx \exp(ig\epsilon A_{\mu}(x)n^{\mu}) \\ &\approx 1 + ig\epsilon n^{\mu}A_{\mu}(x) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{3}$$

[4]See [2] for a more complete discussion.

Putting this form for $U(x+\epsilon n, x)$ into the defintion of the covariant derivitive, Equation (2), gives

$$\begin{aligned}
n^\mu D_\mu \chi(x) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\chi(x+\epsilon) - U(x+\epsilon, x)\chi(x)\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\chi(x+\epsilon) - (1 + ig\epsilon n^\mu A_\mu(x))\chi(x) + \mathcal{O}(\epsilon^2)\right] \\
&= \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left[\chi(x+\epsilon) - \chi(x)\right] - ign^\mu A_\mu(x)\chi(x) + \mathcal{O}(\epsilon)\right) \\
&= n^\mu\left[\partial_\mu - igA_\mu(x)\right]\chi(x)
\end{aligned}$$

We have found that the covariant derivitive has to have the form $D_\mu = \partial_\mu - igA_\mu(x)$ where $A_\mu(x)$ is an element of the Lie algebra of $SU(N)$. The transformation properties of $A_\mu(x)$ can be found by subbing Equation (3) into Equation (1) which gives

$$A_\mu(x) \to V(x)A_\mu V(x)^\dagger - \frac{i}{g}(\partial_\mu V(x))V^\dagger(x)$$

This is precisely the right transformation to make the covariant derivitive transform as required

$$D_\mu\chi(x) \to V(x)D_\mu\chi(x)$$

And so finally, after all this work, we have a Lagrangain which is invariant under local transformations of the fields. In general, $A_\mu(x)$ can be dynamical and so we want a kinetic term for it. This should be gauge invariant also. $A_\mu(x)$ is an element of the Lie algebra of $SU(N)$ and so can be expanded in a basis of it: $A_\mu(x) = A^a_\mu(x)T^a$ where T^a are the basis elements of the Lie algebra and $A^a_\mu(x)$ are the **gauge fields**. It is important to realise that $A^a_\mu(x)$ are field co-efficients and T^a are a basis (of matrices) for $SU(N)$. Let $F_{\mu\nu}$ denote the field strength which is also in the Lie algebra. We can expand it in a basis of the Lie algebra so that $F_{\mu\nu} = F^a_{\mu\nu}T^a$. The field strength is defined by

$$\begin{aligned}
F_{\mu\nu} &= [D_\mu, D_\nu] \qquad \text{so that} \\
F_{\mu\nu}\Psi &= [D_\mu, D_\nu]\Psi \\
&\to [D_\mu, D_\nu](U(x)\Psi) \qquad \text{under a gauge transformation} \\
&= U(x)[D_\mu, D_\nu]\Psi \qquad \text{by the definition of covariant derivitive} \\
&= U(x)[D_\mu, D_\nu]U(x)^{-1}(U(x)\Psi)
\end{aligned}$$

We see that $F_{\mu\nu} \to U(x)F_{\mu\nu}U(x)^{-1}$. For the kinetic term

$$\begin{aligned}
F^a_{\mu\nu}F^{\mu\nu,a} &= 2F^a_{\mu\nu}F^{\mu\nu,b}\frac{\delta^{ab}}{2} \\
&= 2F^a_{\mu\nu}F^{\mu\nu,b}\mathrm{tr}(T^aT^b) \\
&= 2\mathrm{tr}(F^a_{\mu\nu}T^aF^{\mu\nu,b}T^b) \\
&= 2\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \\
&\to 2\mathrm{tr}(U(x)F_{\mu\nu}U(x)^{-1}U(x)F^{\mu\nu}U(x)^{-1}) \\
&= 2\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \qquad \text{by cyclicity of trace}
\end{aligned}$$

which is gauge invariant. The generators for the representation of $SU(N)$ are normalised such that:

$$\mathrm{Tr}_R(T^aT^b) = T(R)\delta^{ab}$$

where $T(R)$ is the Dynkin index of the representation. Notation in the literature for the trace varies, with Tr commonly meaning the trace is performed in the adjoint representation, while tr is used for the trace on the fundamental representation. For a trace on a general $R-$dimensional representation, Tr_R is commonly used. $T(R)$ is $1/2$ for the fundamental representation and N for the adjoint representation of $SU(N)$. Above we have used $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$ so we have assumed that the matter fields are in the fundamental representation of $SU(N)$ (which they normally are in the standard model, only the right handed fields in the electroweak theory are not, see my notes on representations). But we could have used $\mathrm{Tr}_R(T^aT^b) = T(R)\delta^{ab}$ to show that the kinetic term is invariant under gauge transformations when the gauge fields are in a general representation of $SU(N)$.

We have the final form for a Lagrangian that is invariant under a gauge transformation:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\chi}(x)(i\gamma^\mu D_\mu - m)\chi(x)$$

3 Some observations

- A mass term of the form $m^2 A^a_\mu A^a_\mu$ is not gauge invariant. So including a mass term for the gauge boson would break gauge invariance. This requires that the gauge boson must be **massless**. However, gauge bosons can acquire a mass through spontaneous symmetry breaking, e.g, the $W^\pm, Z^0$.

- $A_\mu(x)$ is an element of the Lie algebra of $SU(N)$, the gauge group in this case. As such, $A_\mu(x)$ can be expanded in a basis of the Lie algebra: $A_\mu(x) = A^a_\mu(x)T^a$ where $A^a_\mu(x)$ are the gauge fields coefficients to the (matrix) generators T^a. There are $dim(SU(N))$ gauge fields.

- We can generalise to the case where the gauge group is some compact group $\mathcal{G}$. Then there will be $dim(\mathcal{G})$ gauge fields.

- In non abelian gauge theories, $F^a_{\mu\nu}$ contains a term of the form $f^a_{bc} A^b_\mu A^c_\nu$. So the term $F^a F^a$ in the lagrangian will contain A^3 and A^4 terms. These are the **self interactions** of the gauge fields. These are described in [1] in the section on gauge theories.

- One of the Clay problems is to find the lightest non-zero energy state in a pure Yang Mills gauge theory (a gauge theory with no matter fields) in $\mathbb{R}^4$. The solution should also explain confinement of quarks.

4 Extension of what we have done so far

We have shown how to construct a gauge theory when the Lagrangian transforms under $SU(N)$. In general, the Lagrangian could transform under some **representation**[5] of $SU(N)$. So far, we have described how to build gauge theories for the fundamental representation of the gauge group. If the Lagrangian transforms under some representation of $SU(N)$ then the matter fields $\chi(x)$ must be in a representation space of $SU(N)$, on which the representation acts. When the Lagrangian transforms under some representation of the gauge group, we can construct the covariant derivitive in the same way as before. It has the form

$$D_\mu = \partial_\mu - igA_\mu(x)$$

where $A_\mu(x)$ is an element of the representation of the Lie algebra. We can expand $A_\mu(x)$ in a basis of the representation so that $A_\mu(x) = A^a_\mu(x)T^a$, where $A^a_\mu(x)$ are the gauge field coefficients and T^a are the (matrix) generators for the representation. All the conclusions that we drew previously when working in the fundamental representation are the same when we work in other representations of the gauge group. The only difference is that T^a are now generators for the representation.

D_μ is defined to transform under an element of the representation, V, of the gauge group as

$$D_\mu \chi \to V D_\mu \chi = (V D_\mu V^{-1})(V\chi)$$

and so $D_\mu \to V D_\mu V^{-1}$. The covariant derivative is said to transform adjointly because of this transformation property. The gauge fields $A^a_\mu(x)$ are said to transform adjointly in order to make the covariant derivitive transform correctly. The generators do not have to be in the adjoint representation of the group and in general they are not. This is just standard (and sometimes confusing) terminolgy.

See the last section in the representaion notes on my webpage to see how the different fields in transform under the standard model gauge group.

5 References

[1] Prof. Hugh Osborn's AQFT notes.

[2] MIT free online courses available here.

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[5] See the notes on Representations on my webpage