# The Schwarzschild Metric

Relativity and Astrophysics
Lecture 34
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## Outline

- Schwarzschild metric
  - Spatial part
  - Time part
- Coordinate Frames
  - Free-float
  - Shell
  - Schwarzschild bookkeeper
- Principle of Extremal Aging
  - Conservation of Energy

# Schwarzschild Metric

- In the presence of a spherically symmetric massive body, the flat spacetime metric is modified.
- This metric is found by solving Einstein's field equations for general relativity
  - Karl Schwarzschild found the solution within a month of the publication of Einstein's general theory of relativity
  - This metric is the solution for curved empty spacetime on a spatial plane through the center of a spherically symmetric (nonrotating) center of gravitational attraction
- The timelike form of the solution is:

$$d\tau^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$

- where
  - $\phi$ = angular coordinate with the same meaning as in Euclidean geometry
  - $r$ = the reduced circumference.
  - $t$ = the "far-away" time which is measured on clocks far away from the center of attraction.

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# Schwarzschild Metric (cont'd)

- The spacelike form of the solution is:

$$d\sigma^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\phi^2$$

- Every (non-quantum) feature of spacetime around a spherically symmetric non-rotating uncharged massive body is described by the Schwarzschild metric.
  - Note – clearly something strange happens at $r = 2M$ (the Schwarzschild radius or event-horizon radius) – more later
- Why this looks right
  - The curvature factor $(1 - 2M/r)$ appears in the $dt$ and $dr$ terms and depends on $r$ but not $\phi$. This is expected for spherical symmetry.
  - As $r$ gets very large, the metric becomes flat (spacetime)
  - As $M$ becomes small, the metric becomes flat (no curvature without mass)

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# Spatial part of metric

- For fixed $t$, we have the spatial part of the metric

$$d\sigma^2 = \frac{dr}{1-2M/r} + r^2 d\phi^2 \qquad dt = 0$$

- The factor for the $dr^2$ term is $1/(1 - 2M/r)$ which is greater than one for $r > 2M$, thus for fixed $\phi$, $d\sigma > dr$.
  - Think of a rod extending directly ($d\phi = 0$) between to concentric shells. Two firecrackers go of at each end at the same time, $dt = 0$. The proper distance between the explosions is

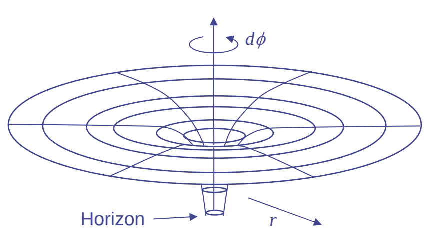
$$d\sigma = dr_{shell} = \frac{dr}{(1-2M/r)^{1/2}} \qquad dt = 0,\ d\phi = 0$$

  - This says that the distance between shells is greater that the difference in $r$-values.
  - Thus we have spatial curvature.

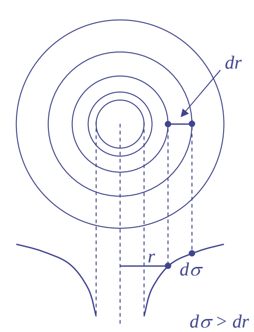


# "Picturing" the Space Part

- Using an "embedding diagram" we can attempt to visualize the spatial part of the Schwarzschild metric
  - The limitation here is that is that the vertical dimension is <u>not</u> an extra dimension of space
  - The only the parabolic surface represents curved-space geometry (you have to stay on the surface – since locations off the surface don't exit!)




Spacetime geometry for a plane sliced through the center of a black hole



# Time part of metric

- For fixed $r$ and $\phi$, the metric is

$$d\tau = dt_{shell} = (1 - 2M/r)^{1/2} dt$$

Clock at rest on a shell at radius r

  - The $dt$ term is the "far-away" time (ephemeris time) and $d\tau$ is the proper time (tick occur on the same clock).
- Time Dilation
  - Consider two successive ticks (events) of a clock on a shell.
  - The factor for the $dt$ term is $(1 - 2M/r)$ which is less than one so that $d\tau < dt$.
  - This means that the time between ticks ($dt_{shell}$) is smaller at emission than their value ($dt$) when received at a great distance.
    - Send out a light pulse from $r_{shell}$ to a distant observer with each clock tick.
    - The remote observer will measure a longer time ($dt$) between ticks than $dt_{shell}$.



# Gravitational Redshift

- Instead of using pulses of light, use the light wave itself.
  - The period of the light wave is a form of ticking. A the distant observer measures a longer time between ticks (wave crests) than someone at $r_{shell}$.
- The light is red-shifted (longer time between crests => lower frequency) to longer a wavelength
  - Switching the originator & receiver gives a gravitational blue shift.
- Example: redshift between two radii
  - Suppose light is emitted at $r_1 = 4M$ and absorbed at $r_2 = 8M$. By what fraction is the period of the light increased?

$$\frac{dt_{shell,1}}{(1-2M/r_1)^{1/2}} = dt = \frac{dt_{shell,2}}{(1-2M/r_2)^{1/2}} \quad \Rightarrow \quad \frac{dt_{shell,2}}{dt_{shell,1}} = \frac{(1-2M/r_2)^{1/2}}{(1-2M/r_1)^{1/2}}$$

  - The ratio of the periods is then

$$\frac{dt_{shell,2}}{dt_{shell,1}} = \frac{(1-1/4)^{1/2}}{(1-1/2)^{1/2}} = \frac{0.866}{0.707} = 1.22$$

  - So that the wavelength shifts long ward (to the red) by a factor of 1.22 as the photon climbs from $r = 4M$ to $r = 8M$, e.g. yellow goes to red.

# Choice of Reference Frames

- There a number of reference frames we can choose from to examine the Schwarzschild metric. For instance,
  - A free-fall (free-float) frame
  - Standing on a shell at given radius
  - Use $r$, $\phi$, and far-away time $t$ (Schwarzschild bookkeeping)
- Each of these observations requires a different set of coordinates and provides a different way of examining spacetime around a black hole.
  - Can a person exist in these frames?
  - If so, what kind of existence is it?
  - How do we relate one frame to another?



# Free-float frame

- Consider an unpowered spaceship falling into a black hole
  - As measured by "shell" observers the ship increases in speed as the ship plunges in.
  - Inside the ship, we have a special-relativity capsule equivalent to being in open space
- However, this frame is only local as tides will produce accelerations between separated particles => it is not a free-float (inertial) reference frame
  - Need to make the spatial and temporal extent of the frame smaller (easy near the Earth or far from a gravitating body)
  - But the center of a black hole tides become very strong, able to rip apart any physical object
- This frame is the only one in which humans could exist near a black hole.
  - For a large enough black hole – tidal forces could be tolerated by humans.

# Local Shell Frame

- We live on a (nearly) spherical shell – Earth's surface
  - This surface forces us away from the natural motion of a free particle
  - This is the "force of gravity" we experience which is direct toward the center of the Earth
- What is the form of the metric for a shell observer? We have

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\phi^2$$

- And substituting our previous expression for dr and dt

$$dr_{shell} = dr \big/ (1 - 2M/r)^{1/2} \qquad d\tau = dt_{shell} = (1 - 2M/r)^{1/2}\, dt$$

  - gives

$$d\tau^2 = dt_{shell}^2 - dr_{shell}^2 - r^2 d\phi^2$$

- which looks like flat spacetime but it is not since $dt$, $dr$ and $rd\phi$ are all functions of $r$. Notice that this gives $dt$ = 0 for light on the shell since the distance along the surface is given by $ds_{shell}^2 = dr_{shell}^2 + r^2 d\phi^2$

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# Local Shell Frame (cont'd)

- This frame has to deal with the gravitational force (since it is not a free float frame)
  - Special relativity (SR) describes brief, local experiments
- Shell vs. free-float frame
  - SR will work well for a longer time in a free-float frame by making the spatial extent smaller (always have gravity in shell frame)
  - Free-float observer can cross the horizon and continue experiments (until tidal forces rip her apart)
    - Inside the horizon neither shell nor shell observers can exist
- Shell and free-float observers can compare local measurements using special relativity (including the Lorentz transformation)

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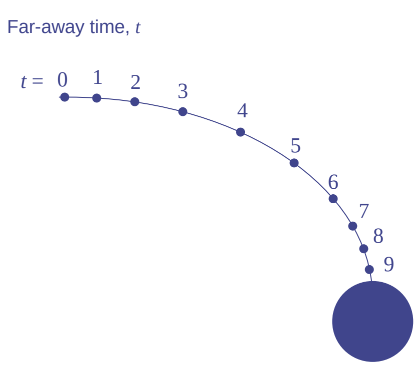
# Schwarzschild coordinates

- The Schwarzschild coordinates $t$, $r$, and $\phi$ provide a global description of events
  - These events can be far apart
- The Schwarzschild observer is a bookkeeper who rarely makes measurements herself
  - She examine reports from local shell and free-float observers and combines these to describe events that span spacetime around the black hole
  - The local observers convert there coordinates to Schwarzschild coordinates and send them back to the Schwarzschild bookkeeper
- No one “lives” in these coordinates. They are an accounting system
  - Bookkeeper coordinates have universality but most of the data entries are isolated from direct experience



# Trajectories




- Schwarzschild map
  - Table kept by bookkeeper of coordinates sent to her by shell (or free float) observers along the trajectory of a satellite
- No one directly observes this trajectory
  - This is not the way the orbit would look to an observer
  - There are time delays and gravitational bending of light
- The dots are closer together at the beginning and end of the trajectory
  - Why?

# Schwarzschild lattice & observer

- Given knowledge of the metric, we can construct a set of lattice points with clocks akin to what we did for flat spacetime
  - Clocks are placed at the shell coordinates and $r$, and $\phi$ are stamped at each point
  - Shell and free-fall observers can read off these coordinates and the clock reading for events
- Why use local observers?
  - Not required, but allows us to use local physics (e.g. atomic clock runs just fine locally)
  - Or "plunge" into a black hole



# Applicability

- The Schwarzschild metric applies only outside the surface of an object
  - Also, slowly spinning object like the Earth and Sun are okay
  - A black hole has no "surface" so the Schwarzschild metric can apply to almost $r = 0$ (the singularity).