11 | Local Gauge Invariance

Quantum electrodynamics stands out as a fundamental theory of electromagnetism that incorporates the principles of quantum mechanics and special relativity in a consistent way. A conspicuous ingredient of the theory, whose classical formulation was first given by Maxwell (1864), is its invariance under local gauge transformations. Under these transformations wave functions (or fields) associated with charged particles change their phase by an amount that may vary from one space-time point to another. It is possible to generalize these gauge transformations to more complicated groups of transformations, leading to a large class of so-called *gauge theories*. Experimentally, it is now a well-established fact that these theories underly the strong, weak and electromagnetic interactions between elementary particles, as we shall discuss in due course. We should point out here that also the gravitational force is described by a gauge theory, as the theory of general relativity is invariant under continuous reparametrizations of space-time, which are called general coordinate transformations. However, general coordinate transformations are of a different nature than the generalized local phase transformations and fall outside the framework of this book. In this chapter we will start by examining the immediate consequences of local gauge invariance, and discuss the principal concepts that are involved.

11.1. Local versus rigid invariance

Let us consider a dynamical system described in terms of a complex function $\psi(x)$ of the space-time coordinates, whose time evolution is governed by equations that are invariant under arbitrary phase transformations

$$\psi(x) \to \psi'(x) = \mathrm{e}^{\mathrm{i}\xi}\,\psi(x)\,. \tag{11.1}$$

Here ξ is the (constant) transformation parameter that determines the amount by which the phase of ψ is changed. For example, $\psi(x)$ could be the wave function of a nonrelativistic electron, which must satisfy the Schrödinger equation. If the dynamics is invariant under (11.1) it is entirely a matter of convention how one chooses the phase of $\psi(x)$, because both ψ and ψ' are solutions of the same equation. However, once one has made a choice, say by specifying the initial value of the wave functions at some given time t, then the equations that govern the time evolution of ψ will determine the value of the phase at any other instant of time, so that the theory only determines phase *differences*. The overall phase may only be meaningful in that it reflects the fact that we are dealing with a degenerate situation, where functions $\psi(x)$ with the same modulus but different phase describe completely similar but independent realizations of a physical system. In the particular case where $\psi(x)$ is some (properly normalized) quantum-mechanical wave function, the overall phase loses its physical significance because the physical content of a wave function resides in its modulus, which defines the probability for a system to be at a corresponding position. Evidently the consistency of the quantum-mechanical probability interpretation requires invariance under the phase transformations (11.1) in order to guarantee the existence of a conserved probability current (see, for example section 3.1).

Suppose now that one would insist on having the freedom of choosing the phase of ψ at each point in space-time separately. Obviously in that case the theory in question must be invariant under

$$\psi(x) \to \psi'(x) = \mathrm{e}^{\mathrm{i}\xi(x)}\,\psi(x)\,, \tag{11.2}$$

where the transformation parameter ξ is now an arbitrary (differentiable) function of space-time coordinates. Such transformations are called *local* gauge transformations, while constant transformations such as (11.1) are called *rigid* or *global* transformations. Obviously, if a theory is invariant under (11.2) then it must be entirely independent of the phase of ψ. Phase differences have therefore no physical significance either and are no longer determined by the dynamical equations of the theory. Strictly speaking, the initial value problem for these theories is therefore not well-defined; if we start from an initial configuration and calculate ψ at some later time, then its phase is left unspecified in view of the fact that the invariance under local gauge transformations still allows us to change the phase at that time. This fact may lead to technical difficulties in calculations, which can be avoided by first selecting explicit values for the phase at every point. Such a choice is called a *gauge condition*. In section 4.2 we already noted the necessity for employing a gauge condition when deriving the propagator for a massless vector field. As we emphasized there, the physical results should not be affected by the choice of the gauge condition, because one has only imposed restrictions on degrees of freedom which are not present in the gauge invariant theory that was the original starting point. In other words, physical quantities should be *gauge independent*.

By insisting on the freedom to change fields by an arbitrary group transformation at each space-time point separately, it turns out that one needs a quantity that can carry information regarding these transformations from one space-time point to another. This quantity is the *gauge field*; its existence can be viewed as a prime consequence of choosing a framework based on local gauge invariance, as will be demonstrated in the subsequent sections.

Although rigid and local transformations such as (11.1) and (11.2) look very similar, we should emphasize that they have entirely different physical consequences. Invariance under rigid transformations implies that certain physical degrees of freedom are degenerate, i.e. they behave in an identical way. On the other hand the invariance under local transformations implies that some of the degrees of freedom are simply absent, a fact to which no physical significance can be attributed. The reader may now start wondering about the relevance of local gauge invariance: if a theory does not depend on certain degrees of freedom, in the case at hand those associated with the phase of ψ, then why not just drop this phase and choose ψ real from the beginning? In principle, this is a valid point of view. However, in practice it is often cumbersome to decompose the dynamical variables in such a way that the gauge degrees of freedom decouple. Furthermore, one can show that a manifestly Lorentz invariant and local description of massless particles with spin $s \geqslant 1$ contains gauge invariance as a necessary ingredient. In fact, as we will see later in this chapter, there is a subtle connection between Lorentz invariance, gauge invariance and the fact that massless vector particles couple to conserved currents. In addition, there may be global obstructions to remove the gauge degrees of freedom everywhere in space-time. For instance, in the Aharonov-Bohm effect described in section 1.3 we found that although the vector potential can *locally* be put to zero by means of a suitable gauge transformation, this is not possible *globally* around a closed curve which encloses a confined magnetic flux. Another complication occurs when ψ vanishes somewhere so that its phase will become ill-defined. In that case a description in terms of a real quantity ψ will become very unpractical.

11.2. Gauge field theory of U*(1)*

As a first example let us construct a field theory which is invariant under local phase transformations. Our starting point is the free Dirac Lagrangian

$$\mathcal{L}_\psi = -\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi\,, \tag{11.3}$$

which is obviously invariant under rigid phase transformations, because

$$\psi \to \psi' = \mathrm{e}^{\mathrm{i}q\xi}\,\psi, \tag{11.4}$$

implies

$$\bar{\psi} \to \bar{\psi}' = \mathrm{e}^{-\mathrm{i}q\xi}\,\bar{\psi}\,. \tag{11.5}$$

Here we have introduced a parameter q that measures the strength of the phase transformations, because eventually we want to consider fields transforming with different strengths. Phase transformations generate the group of 1×1 unitary matrices called U(1).

Let us now consider local phase transformations, and verify whether (11.3) remains invariant. Of course, the local aspect of the transformation is not important for the invariance of the mass term, since the variation of that term only involves the transformation of fields taken at the same point in space-time. But as soon as we compare fields at different points in space-time the local character of the transformation is crucial. A derivative, which depends on the variation of the fields in an infinitesimally small neighbourhood, will be subject to transformations at neighbouring space-time points. To see the effect of this, let us evaluate the effect of a local transformation on $\partial_\mu\psi$,

$$\begin{aligned} \partial_\mu\psi(x) \to \big(\partial_\mu\psi(x)\big)' &= \partial_\mu\big(\,\mathrm{e}^{\mathrm{i}q\xi(x)}\,\psi(x)\big) \\ &= \mathrm{e}^{\mathrm{i}q\xi(x)}\,\big(\partial_\mu\psi(x) + \mathrm{i}q\,\partial_\mu\xi(x)\,\psi(x)\big)\,. \end{aligned} \tag{11.6}$$

Clearly $\partial_\mu\psi$ does not have the same transformation rule as ψ itself. There is an extra term induced by the transformations at neighbouring space-time points which is proportional to the derivative of the transformation parameter. This term is responsible for the lack of invariance of the Lagrangian (11.3).

In order to make (11.3) invariant under local phase transformations, one may consider the addition of new terms whose variation will compensate for the $\partial_\mu\xi$ term in (11.6). As a first step in this direction one could attempt to construct a modified derivative D_μ transforming according to

$$D_\mu\psi(x) \to (D_\mu\psi(x))' = \mathrm{e}^{\mathrm{i}q\xi(x)}(D_\mu\psi(x))\,. \tag{11.7}$$

If such a derivative exists we can then simply replace the ordinary derivative ∂_μ in the Lagrangian (11.3) by D_μ and establish invariance under *local* phase transformations.

Since the transformation of $D_\mu\psi$ is entirely determined by the transformation parameter at the same space-time coordinate as ψ, D_μ is called a *covariant derivative*. To appreciate this definition one should realize that a local phase transformation may be regarded as a product of independent phase transformations each acting at a separate space-time point. It is possible that local quantities transform only under the gauge transformation taken at the same space-time point. Such quantities are then said to transform *covariantly*. For instance, according to this nomenclature, the field ψ transforms in a covariant fashion under local phase transformations, whereas the transformation behaviour of ordinary derivatives (cf. 11.6), although correctly representing the action of the full local group, is clearly noncovariant. It is convenient to have local quantities transforming covariantly, and this is an extra motivation for introducing the covariant derivative.

Let us now turn to an explicit construction of the covariant derivative. Comparing (11.6) and (11.7) we note that the modified derivative D_μ must contain a quantity whose transformation can compensate for the second term in (11.6). If we define

$$D_\mu\psi(x) = (\partial_\mu - \mathrm{i}qA_\mu(x))\,\psi(x)\,, \tag{11.8}$$

we obtain

$$\begin{aligned} D_\mu\psi \to (D_\mu\psi)' &= (\partial_\mu\psi)' - \mathrm{i}q(A_\mu\psi)' \\ &= \mathrm{e}^{\mathrm{i}q\xi}(\partial_\mu\psi + \mathrm{i}q\,\partial_\mu\xi\,\psi - \mathrm{i}qA'_\mu\psi)\,. \end{aligned} \tag{11.9}$$

Comparing this to (11.7) shows that the new quantity A_μ must have the following transformation rule

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\xi\,. \tag{11.10}$$

Hence the requirement of local gauge invariance has led us to introduce a new field A_μ, whose transformation is given by (11.10). This new field is called a *gauge field*. Note that the gauge field does *not* transform in a covariant fashion.

Introducing the covariant derivative (11.8) into the Lagrangian (11.3) shows that the theory is no longer free, but describes interactions of the fermions with the gauge field

$$\begin{aligned} \mathcal{L}_\psi &= -\bar\psi \not{D}\psi - m\bar\psi\psi, \\ &= -\bar\psi\not{\partial}\psi - m\bar\psi\psi + \mathrm{i}qA_\mu\bar\psi\gamma^\mu\psi\,. \end{aligned} \tag{11.11}$$

Usually one assumes that A_μ describes some new and independent degrees of freedom of the system, although this can sometimes be avoided. But in any case it is clear that the requirement of local gauge invariance leads to interacting field theories of a particular structure.

Covariant derivatives play an important role in theories with local gauge invariance, so we discuss them here in more detail. First we note that D_μ consists of two terms which are both related to an infinitesimal transformation. The derivative $\partial_\mu\psi$ can be interpreted as the result of an infinitesimal space-time displacement on ψ, while the second term $-\mathrm{i}qA_\mu\psi$ represents the result of an infinitesimal gauge transformation on ψ. More precisely, the infinitesimal displacement of the coordinates, $x^\mu \to x^\mu + a^\mu$, accompanied by a field-dependent gauge transformation with parameter $\xi = -a^\mu A_\mu$, leads to a so-called *covariant translation*, generated by (cf. problem 11.2)

$$\delta\psi = a^\mu D_\mu\psi\,. \tag{11.12}$$

The observation that D_μ corresponds to an infinitesimal variation shows that covariant derivatives must satisfy the Leibniz rule, just as ordinary derivatives do

$$D_\mu(\psi_1\psi_2) = (D_\mu\psi_1)\psi_2 + \psi_1(D_\mu\psi_2)\,. \tag{11.13}$$

To appreciate this result one should realize that the precise form of the covariant derivative is tied to the transformation character of the quantity on which it acts. For instance, if ψ_1 and ψ_2 transform under local phase transformations with strength q_1 and q_2, respectively, then we have

$$\begin{aligned} D_\mu(\psi_1\psi_2) &= (\partial_\mu - \mathrm{i}(q_1+q_2)A_\mu)(\psi_1\psi_2)\,, \\ D_\mu\psi_1 &= (\partial_\mu - \mathrm{i}q_1 A_\mu)\psi_1\,, \\ D_\mu\psi_2 &= (\partial_\mu - \mathrm{i}q_2 A_\mu)\psi_2\,. \end{aligned} \tag{11.14}$$

With these definitions it is straightforward to verify the validity of (11.13).

Of course, repeated application of covariant derivatives will always yield covariant quantities. This fact may be used to construct a new covariant object which depends only on the gauge field. For instance, let us apply the antisymmetric product of two derivatives on ψ

$$[D_\mu, D_\nu]\psi = D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi)\,. \tag{11.15}$$

Writing explicitly

$$D_\mu(D_\nu\psi) = \partial_\mu\partial_\nu\psi - \mathrm{i}qA_\mu\,\partial_\nu\psi - \mathrm{i}q(\partial_\mu A_\nu)\psi - \mathrm{i}qA_\nu\,\partial_\mu\psi - q^2 A_\mu A_\nu\psi\,, \tag{11.16}$$

one easily establishes

$$[D_\mu, D_\nu]\psi = -\mathrm{i}qF_{\mu\nu}\psi\,, \tag{11.17}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{11.18}$$

However, since ψ transforms covariantly and the left-hand side of (11.17) is covariant, we conclude that $F_{\mu\nu}$ is itself a covariant object. In fact , application of the gauge transformation (11.10) shows that $F_{\mu\nu}$ is even gauge *invariant*,

$$\delta F_{\mu\nu} = \partial_\mu\partial_\nu\xi - \partial_\nu\partial_\mu\xi = 0\,, \tag{11.19}$$

but, as we will appreciate later on, this is a coincidence related to the fact that U(1) transformations are abelian.

The result (11.17) is called the *Ricci identity*. It specifies that the commutator of two covariant derivatives is an infinitesimal gauge transformation with parameter $\xi = -F_{\mu\nu}$, where $F_{\mu\nu}$ is called the field-strength. This field strength $F_{\mu\nu}$ is sometimes called the *curvature* tensor. The reason for this nomenclature is not difficult to see: if the left-hand side of (11.17) were zero then two successive infinitesimal covariant translations, one in the $\hat{\mu}$ and the other in the $\hat{\nu}$ direction, would lead to the same result when applied in the opposite order. According to the Ricci identity this is not the case for finite $F_{\mu\nu}$. One encounters the same situation when considering translations on a curved surface, which do not commute for finite curvature. As the tensor $F_{\mu\nu}$ on the right-hand side of (11.17) measures the lack of commutativity, its effect is analogous to that of curvature.

We can use covariant derivatives to obtain yet another important identity. Consider the double commutators of covariant derivatives, $[D_\mu,[D_\nu,D_\rho]]$. Writing out all the twelve terms explicitly one can easily verify that the cyclic combination vanishes identically,

$$[D_\mu,[D_\nu,D_\rho]]+[D_\nu,[D_\rho,D_\mu]]+[D_\rho,[D_\mu,D_\nu]]=0\,. \tag{11.20}$$

This identity is known as the Jacobi identity. To see its consequences let us write the first term acting explicitly on $\psi(x)$,

$$\begin{aligned}[D_\mu,[D_\nu,D_\rho]]\psi &= D_\mu([D_\nu,D_\rho]\psi)-[D_\nu,D_\rho]D_\mu\psi \\ &= -\mathrm{i}q\,(D_\mu F_{\nu\rho})\psi\,,\end{aligned} \tag{11.21}$$

where we used (11.13) and (11.17). Therefore the Jacobi identity implies

$$D_\mu F_{\nu\rho}+D_\nu F_{\rho\mu}+D_\rho F_{\mu\nu}=0\,. \tag{11.22}$$

In this case the field strength is invariant under gauge transformations so we may replace covariant by ordinary derivatives and obtain

$$\partial_\mu F_{\nu\rho}+\partial_\nu F_{\rho\mu}+\partial_\rho F_{\mu\nu}=0\,. \tag{11.23}$$

This result is called the *Bianchi identity*; it implies that $F_{\mu\nu}$ can be expressed in terms of a vector field, precisely in accord with (11.18). In four dimensions the Bianchi identity is often written as

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}=0\,. \tag{11.24}$$

The field strength tensor can be used to write a gauge invariant Lagrangian for the gauge field itself,

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}^2\,. \tag{11.25}$$

This Lagrangian can now be combined with (11.11),

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_A+\mathcal{L}_\psi \\ &= -\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2-\bar\psi\not\partial\psi-m\bar\psi\psi+\mathrm{i}qA_\mu\bar\psi\gamma^\mu\psi\,.\end{aligned} \tag{11.26}$$

so that we have obtained an interacting theory of a vector field and a fermion field invariant under the combined local gauge transformations (11.2) and

(11.10). It is not difficult to see that this theory coincides with electrodynamics: the gauge field A_μ is just the vector potential (subject to its familiar gauge transformation), which couples to the fermion field via the minimal substitution, and $F_{\mu\nu}$ is the electromagnetic field strength.

To derive the field equations corresponding to (11.26) is straightforward. They read

$$\begin{aligned} \partial^\nu F_{\mu\nu} &= \mathrm{i}q\bar{\psi}\gamma_\mu\psi\,, && (11.27)\\ (\not{\partial}+m)\psi &= \mathrm{i}q\not{A}\psi && (11.28)\\ \bar{\psi}(\overleftarrow{\not{\partial}}-m) &= -\mathrm{i}q\bar{\psi}\not{A}\,, && (11.29) \end{aligned}$$

where the right-hand side of (11.27), which originates from the coupling of the vector potential to the fermion fields, follows from the infinitesimal variation of $\mathcal{L}_\psi$ with respect to A_μ. If $\mathcal{L}_\psi$ does not depend on derivatives of A_μ, this variation defines a four-vector current $J_\mu = \partial\mathcal{L}_\psi/\partial A_\mu$; when this is not the case one uses a slightly more general definition,

$$\delta\Big(\int \mathrm{d}^4x\,\mathcal{L}_\psi(\psi,A)\Big) = \int \mathrm{d}^4x\;\delta A_\mu(x)\,J^\mu(x)\,, \qquad (11.30)$$

where we assume that the variation δA_μ vanishes at the boundary of the space-time integration domain so that we can freely perform integrations by parts to remove all derivatives on δA_μ on the right-hand side of (11.30). We will often make a distinction between "matter" and gauge fields and use the above definition of the current in more general situations with ψ generically denoting all "matter" fields. For the case at hand we have

$$J_\mu = \mathrm{i}q\bar{\psi}\gamma_\mu\psi. \qquad (11.31)$$

Clearly (11.27) corresponds to the inhomogeneous Maxwell equation (1.66), while the Bianchi identity (11.24) coincides with the homogeneous Maxwell equation (1.65). The field equations (11.28) and (11.29) describe the dynamics of the charged fermion, and are strictly speaking not part of Maxwell's equations.

It is easy to repeat the above construction for other fields. For example, a complex scalar field ϕ may transform under local phase transformations according to

$$\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\xi(x)}\,\phi(x)\,. \qquad (11.32)$$

As before, the requirement of local gauge invariance forces one to replace the ordinary derivative by a covariant derivative,

$$D_\mu\phi = (\partial_\mu - \mathrm{i}qA_\mu)\phi\,, \qquad (11.33)$$

so that one obtains a gauge invariant version of the Klein-Gordon Lagrangian,

$$\begin{aligned}\mathcal{L}_\phi &= -|D_\mu\phi|^2 - m^2|\phi|^2 \\ &= -|\partial_\mu\phi|^2 - m^2|\phi|^2 - \mathrm{i}qA_\mu\,(\phi^* \overleftrightarrow{\partial}_\mu\,\phi) - q^2A_\mu^2|\phi|^2\,.\end{aligned} \tag{11.34}$$

This is the Lagrangian for *scalar electrodynamics*, which we have been using in chapter 4 to describe electromagnetic pion-pion scattering (see also problems 8.4-10). Unlike in *spinor electrodynamics*, which is defined by the Lagrangian (11.11), there are interaction terms in (11.34) that are quadratic in A_μ. Therefore the corresponding expression for the current (11.30) now depends also on the gauge field A_μ. It reads

$$J_\mu = \mathrm{i}q((D_\mu\phi^*)\phi - \phi^*(D_\mu\phi))\,, \tag{11.35}$$

and appears on the right-hand side of the Maxwell equation (11.27). The field equation of the scalar field can be written as

$$D^\mu D_\mu\phi - m^2\phi = 0\,. \tag{11.36}$$

The parameter q that we have been using to indicate the relative magnitude of the change of phase caused by the gauge transformation, also determines the strength of the interaction with the gauge field A_μ. Hence in electromagnetism the particles described by the various fields carry electric charges $\pm q$.[1] We should point out that the electric charge is an arbitrary parameter, which can be chosen at will for any field. This is a consequence of the fact that phase transformations commute. To see this consider two independent group transformations, g_A and g_B. Because g_A and g_B are elements of the group their product $g_A\cdot g_B = g_C$ must again be a group element. Suppose we consider another field for which the strength of all transformations differs uniformly by some factor, say 2. The effect of such a rescaling is to change all group elements g into g^2. One is allowed to change the strength of all group transformations uniformly if the multiplication property of the group remains unchanged, hence if

$$g_A^2\cdot g_B^2 = g_C^2 = (g_A\cdot g_B)^2\,. \tag{11.37}$$

This is only the case if g_A and g_B commute, i.e., if $g_A\cdot g_B = g_B\cdot g_A$. Groups whose elements commute are called *abelian*. For *nonabelian* groups not all elements commute. Therefore the strength of nonabelian transformations are fixed (up to an overall constant) and independent of the representation in which the group acts. Consequently the coupling constants of nonabelian gauge fields are not arbitrary but uniquely determined for each representation. Of course, a nonabelian group may contain invariant subgroups to which this

[1] We shall give a more precise definition of the electric charge in section 11.4.

argument should be applied separately. Indeed, strictly speaking the above conclusion is only correct for simple groups (for a more rigorous exposition of the group-theoretical aspects the reader is advised to consult appendix C).

11.3. Current conservation

The four-vector current J_μ that appears on the right-hand side of the inhomogeneous Maxwell equation must satisfy an obvious restriction. To see this contract (11.27) with ∂^μ and use the fact that $F_{\mu\nu}$ is antisymmetric in μ and ν. It then follows that J_μ must be conserved, i.e.,

$$\partial_\mu J^\mu = 0\,. \tag{11.38}$$

Another way to derive the same result is to make use of the matter field equations. For instance, for a fermion field one has

$$\partial_\mu J^\mu = \partial_\mu(\mathrm{i}q\,\bar\psi\gamma^\mu\psi) = \mathrm{i}q\,\bar\psi(\not{D}\psi) + \mathrm{i}q\,(\bar\psi\,\overleftarrow{\not{D}})\psi\,, \tag{11.39}$$

where in the last equation we replaced ordinary by covariant derivatives. As the reader can easily verify, this has no effect because the A_μ-dependent terms cancel in the combined result. Using the equations of motion (11.28) and (11.29) it follows that (11.39) vanishes. Likewise for charged bosons one can show that the current (11.35) is conserved by virtue of the field equation (11.36) and its complex conjugate.

The underlying reason why using the Maxwell equation or the matter field equations leads to an identical result rests upon the invariance of the Lagrangian under local gauge transformations. To see this let us formally apply an infinitesimal gauge transformation to the action

$$S_\psi[\psi, A] = \int \mathrm{d}^4x\; \mathcal{L}_\psi(\psi, A)\,, \tag{11.40}$$

associated with the matter fields. Clearly the variation of S_ψ originates from both the change of A_μ and of ψ under the gauge transformation. Under a gauge transformation of A_μ the action changes by $\int \mathrm{d}^4x\; \partial_\mu\xi(x)\, J^\mu(x)$, where the current J_μ was generally defined in (11.30). Let us now assume that the gauge parameter $\xi(x)$ vanishes at the boundary of the integration domain used in (11.40), so that the variation of the matter fields will also vanish at the boundary. Therefore an infinitesimal gauge transformation on the matter fields leads to a variation of (11.40) that must be proportional to the matter field equations, as those follow from considering the change of the action under arbitrary variations of the matter fields that vanish at the boundary (cf. the discussion of Hamilton's principle in section 1.1). Hence the change of S_ψ under an infinitesimal gauge transformation takes the form

$$\delta S_\psi = \int \mathrm{d}^4 x \left\{ \partial_\mu \xi(x)\, J^\mu(x) + \text{matter field equations} \right\}, \tag{11.41}$$

and must vanish as S_ψ is gauge invariant. After an integration by parts, observing that (11.41) must vanish for all functions $\xi(x)$ that vanish at the boundary, we conclude that $\partial_\mu J^\mu(x) = 0$ for matter fields satisfying their field equations.

The reader may have noticed that the above derivation is rather similar to the derivation of the Noether theorem, which was presented in section 1.5. Indeed gauge fields couple to the matter fields through a current (cf. 11.30) which is proportional to the Noether current associated with the corresponding rigid transformations (i.e. the transformations with constant $\xi(x)$). In the case at hand, these rigid transformations are

$$\psi \to \mathrm{e}^{\mathrm{i}q\xi}\, \psi\,, \quad A_\mu \to A_\mu\,, \tag{11.42}$$

which clearly constitute an invariance of the action. We now recall that the Noether current follows from replacing ξ by $\xi(x)$. Because A_μ remains inert when making this replacement the action will *not* be invariant under (11.42). Its variation will be proportional to $\partial_\mu \xi(x)$, and it will be multiplied with *minus* the Noether current (cf. **??**). However, the action is gauge invariant, so that the $\partial_\mu \xi(x)$ variation must be precisely cancelled if we let A_μ transform according to (11.10). In order for this cancellation to take place the gauge field must obviously couple to the Noether current. Observe that in this argument we keep the possible A_μ-dependence in the Noether current.

Note, however, that one must be careful in phrasing this result: it is certainly not correct to say that (11.30) is the Noether current associated with the *local* gauge invariance itself, although is is possible to define such a current in principle. Current conservation may be viewed as a reflection of the fact that the degrees of freedom, whose absence is implied by local gauge invariance, do not reappear through the interaction with the matter fields.

The fact that photons must couple to a conserved current has direct consequences for invariant amplitudes that involve external photon lines. Such an amplitude with one photon and several other incoming and outgoing lines takes the form

$$\mathcal{M}(k, \cdots) = \varepsilon_\mu(\mathbf{k})\, \mathcal{M}^\mu(k, \ldots)\,, \tag{11.43}$$

where k and $\varepsilon(\mathbf{k})$ denote the momentum and polarization vector of the photon, respectively. The gauge transformations on the photon field give rise to a term proportional to the photon momentum k_μ, cf. (11.10). To see this let us take the Fourier transform of $A_\mu(x)$ and apply the gauge transformation $\delta A_\mu(x) = \partial_\mu \xi(x)$. Taking the Fourier trasformation, this variation reads $-\mathrm{i}k_\mu \xi(k)$. Gauge invariance thus implies that photon polarizations proportional to k_μ decouple, one expects

$$k_\mu \mathcal{M}^\mu(k, \ldots) = 0\,. \tag{11.44}$$

Obviously this equation takes the same form as (11.38) but now in the momentum representation. However, it turns out that (11.44) holds provided that the external lines (other than that of the photon) are on their mass shell. The latter condition can be understood as a consequence of the fact that we had to impose the matter field equation in (11.41) in order to obtain a conserved current. The decoupling of gauge degrees of freedom is only fully effective when all particles are on their mass shell. These issues will come up again in chapter 13. For the moment we confine ourselves to a discussion of the consequences of current conservation.

Actually, in practice the mass-shell condition for the external lines can be somewhat relaxed and it is usually sufficient when the external lines associated with charged particles are on their mass shell. Therefore it is possible to exploit (11.44) for amplitudes with several off-shell photons, but we caution the reader that this result does not hold for nonabelian gauge theories, as we shall discuss in due course. If all external lines are taken off mass shell one obtains certain relations between Green's functions, which have a more complicated structure than (11.44). Such identities are called *Ward identities.* In the context of quantum electrodynamics these identities are usually called Ward-Takahashi identities. We have already been using them in previous chapters (for a derivation, see problem 8.3).

The fact that A_μ couples to a conserved current is also essential in order to establish that interactions of the massless spin-1 particles associated with A_μ are Lorentz invariant. To explain this observation we recall that massless particles have fewer polarization states than massive ones. This phenomenon was already discussed in section 4.2. For a massive spin-s particle one can always choose to work in the rest frame, where the four-momentum of the particle remains unchanged under spatial rotations, so that its spin degrees of freedom transform according to a $(2s+1)$-dimensional representation of the rotation group SO(3). In other words, there are $2s+1$ polarization states, which transform among themselves under rotations, and which can be distinguished in the standard way by specifying the value of their spin projected along a certain axis. However, for massless particles it is not possible to go to the rest frame and one is forced to restrict oneself to two-dimensional rotations around the direction of motion of the particle. These rotations constitute the group SO(2) (actually, the group of transformations that leave the particle momentum invariant is somewhat larger, but the extra (noncompact) symmetries must act trivially on the particle states in order to avoid infinite-dimensional representations). The group SO(2) has only one-dimensional complex representations. For spin s these representations just involve the states with spin $\pm s$ in the direction of motion of the particle. Consequently massless particles have only two polarization states, irrespective of the value of their spin.

Physical photons have therefore helicity ± 1 and, as we already exhibited in section 4.2, they can be described by transverse polarization vectors $\varepsilon(\mathbf{k})$ that satisfy the conditions

$$k \cdot \varepsilon(\mathbf{k}) = 0\,, \qquad \varepsilon_0(\mathbf{k}) = 0\,, \tag{11.45}$$

which allow indeed two linearly independent polarization vectors.

This immediately raises another question, namely whether the resulting theory can be Lorentz invariant in view of the fact that the second condition in (11.45) is obviously not Lorentz invariant. Nevertheless, it turns out that the interactions of massless spin-1 particles are relativistically invariant, provided these particles couple to conserved currents. To make this more precise, consider a physical process involving a photon, for which the invariant amplitude takes the form

$$\mathcal{M} = \varepsilon_\mu(\mathbf{k})\, \mathcal{M}^\mu(k, ...)\,, \tag{11.46}$$

where $\varepsilon(\mathbf{k})$ is the photon polarization vector, k_μ the photon momentum (with $k^2 = 0$) and the dots indicate the other particle momenta that are relevant. Obviously, (11.46) has a Lorentz invariant form, but since $\varepsilon(\mathbf{k})$ will not remain transverse after a Lorentz transformation, the amplitude will in general no longer coincide with the expression (11.46) when calculated directly in the new frame.

To examine this question in more detail let us first derive how a transverse polarization vector transforms under Lorentz transformations. What we intend to prove is that $\varepsilon_\mu(\mathbf{k})$, satisfying (11.45) with $k^2 = 0$, transforms into a linear combination of a transverse vector $\varepsilon'(\mathbf{k}')$ and the transformed photon momentum k', where $\varepsilon'(\mathbf{k}')$ is transverse with respect to the new momentum k'. More precisely (for a more explicit treatment, see problem 11.1),

$$\varepsilon_\mu(\mathbf{k}) \rightarrow \varepsilon'_\mu(\mathbf{k}') + \alpha\, k'_\mu\,, \tag{11.47}$$

with α some unknown coefficient which depends on the Lorentz transformation. To derive (11.47) it is sufficient to note that the condition $k \cdot \varepsilon(\mathbf{k}) = 0$ is Lorentz invariant, so that the right-hand side of (11.47) should vanish when contracted with k'; therefore it follows that this vector can be decomposed into a transverse vector satisfying (11.45) (but now in the new frame) and the momentum k'. What remains to be shown is that the transverse vector $\varepsilon'(\mathbf{k}')$ has the same normalization as $\varepsilon(\mathbf{k})$. This is indeed the case, since

$$\varepsilon^\mu(\mathbf{k})\, \varepsilon_\mu(\mathbf{k}) \;\; = \;\; \big(\varepsilon'^\mu(\mathbf{k}') + \alpha\, k'^\mu\big)\big(\varepsilon'_\mu(\mathbf{k}') + \alpha\, k'_\mu\big)$$

$$= \quad \varepsilon'^{\mu}(\mathbf{k}')\varepsilon'_{\mu}(\mathbf{k}')\,, \tag{11.48}$$

where we have used (11.45) and $k'^2 = 0$. Using (11.47) one easily establishes that the amplitude (11.46) transforms under Lorentz transformations as

$$\varepsilon_{\mu}(\mathbf{k})\,\mathcal{M}^{\mu}(k,\ldots) \longrightarrow \varepsilon'_{\mu}(\mathbf{k}')\,\mathcal{M}'^{\mu}(k',\ldots) + \alpha\, k'_{\mu}\,\mathcal{M}'^{\mu}(k',\ldots)\,. \tag{11.49}$$

The first term on the right-hand side corresponds precisely to the amplitude that one would calculate in the new frame. Therefore relativistic invariance is ensured provided that the photons couple to a conserved amplitude (cf. 11.44).

11.4. Conserved charges

In classical field theory current conservation implies that the charge associated with the current is locally conserved (cf. 1.52). For scattering and decay reactions of elementary particles it seems obvious that charge conservation implies that the total charge of the incoming particles is equal to the total charge of the outgoing particles. It is the purpose of this section to prove that this is indeed the case. In order to do so it is necessary to give a precise definition of the charge of a particle. In sect. 9.5 this was done in terms of the invariant amplitude for a particle to emit or absorb a zero-frequency photon. To elucidate this definition consider the amplitude for the absorption of a virtual photon with momentum k by a spinless particle. The corresponding diagram is shown in fig 11.1. The momenta of the incoming and outgoing particles are denoted by p and p' respectively, so that $k = p' - p$. Because of Lorentz invariance the amplitude can generally be decomposed into two terms

$$\mathcal{M}_{\mu}(p',p) = F_1(k^2,p^2,p'^2)\,(p'+p)_{\mu} + F_2(k^22,p^2,p'^2)\,k_{\mu}\,, \tag{11.50}$$

where the unknown functions F_1 and F_2 are called *form factors*. We shall assume that the amplitude is correctly normalized according to the prescription discussed in section 3.2 (an explicit example of the way in which the field normalization factors enter is specified in sect. 9.6). We have indicated the most general dependance that the form factors can have. When the charged particles with momenta p and p' refer to physical particles of the same mass, we have $p^2 = p'^2 = -m^2$. In that case, current conservation implies that $k_{\mu}\mathcal{M}^{\mu}(p',p)$ should vanish, so

$$(p'^2 - p^2)\,F_1(k^22,p^2,p'^2) + k^2\,F_2(k^2,,p^2,p'^2) = 0. \tag{11.51}$$

Since the incoming and outgoing particles have the same mass, F_1 drops out from (11.51) and we are left with

$$F_2(k^2,-m^2,-m^2) = 0\,. \tag{11.52}$$

For the on-shell case, for which $p^2 = p'^2 = -m^2$, we can drop the last two arguments in the form factors. In what follows we shall use the form factors in processes where one of the particles is off-shell. However, even in this case we shall only need the the form factors in the on-shell limit. Therefore, from now on we shall only keep the first argument in the form factors.

Observe that $k^2 \geqslant 0$ whenever $p^2 = p'^2$, so that in general we are dealing with an off-shell photon momentum. To obtain (11.52) it is essential that we assume current conservation for *off-shell* photons, because the (on-shell) photon emission by some physical particle would restrict the photon momentum to zero. The assumption of current conservation for off-shell photons can easily be justified, but we refrain from commenting on this aspect here. We should add here that there is an additional reason why F_2 should be absent, as this term violates charge-conjugation symmetry. Charge conjugation has been discussed in sect. 7.1 The function $F_1(k^2)$ is called the *charge form factor*, and as we have been alluding to above, its value at $k^2 = 0$ *defines* the electric charge of the particle in question. For a pointlike particle this is easily verified, and one finds $F_2(k^2) = 0$ and $F_1(k^2) = e$, where e is the coupling constant in the Lagrangian that measures the strength of the photon coupling (see, for instance, sect. 4.3). Experimentally it is not possible to measure the probability for absorbing or emitting a zero-frequency photon, so that the charge of a particle is not measured in this way. It is more feasible to use low-energy Compton scattering (also called Thomson scattering) for this purpose. This process will be discussed in the next section. Another process is the elastic scattering of a particle by a Coulomb field, but in that case one measures only the product of the charge of the particle with the charge associated with the Coulomb field. Moreover the charge is measured by extrapolating to the forward scattering region, where the cross section diverges (see the discussion at the end of sect. 4.3). This makes it hard to extract accurate results.

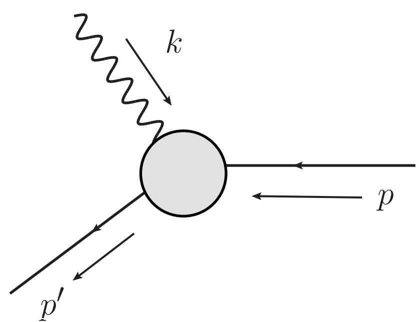

Figure 11.1: The absorption of a virtual photon by a spinless particle.

We will now show that, with the above definition of charge, one has charge conservation in any possible elementary particle reaction. The derivation starts from considering a process in which a soft photon is being emitted or absorbed; for instance

$$A \to B + \gamma\,, \tag{11.53}$$

where A and B denote an arbitrary configuration of incoming and outgoing particles, respectively. It is possible to divide the amplitude for this process into two terms,

$$\mathcal{M}(A \to B + \gamma) = \mathcal{M}^{\rm B}(A \to B + \gamma) + \mathcal{M}^{\rm R}(A \to B + \gamma)\,, \tag{11.54}$$

where $\mathcal{M}^{\rm B}$ consists of all the Born approximation diagrams in which the photon is attached to one of the external lines, as shown in fig 11.2, and $\mathcal{M}^{\rm R}$ represents the remainder. The Born approximation diagrams have the form

$$\begin{aligned} \mathcal{M}^{\rm B} \;\;=\;\; & \varepsilon_\mu(\mathbf{k})\Big\{ \sum_i \mathcal{M}(A[i] \to B)\, \frac{(2p_i - k)^\mu\, F_i(k^2)}{(p_i - k)^2 + m_i^2} \\ & + \sum_j \frac{(2p_j + k)^\mu\, F_j(k^2)}{(p_j + k)^2 + m_j^2}\, \mathcal{M}(A \to B[j]) \Big\}\,, \end{aligned} \tag{11.55}$$

where $\mathcal{M}(A[i] \to B)$ and $\mathcal{M}(A \to B[j])$ denote the invariant amplitude for the process $A \to B$ in which one of the external line momenta is shifted from its mass shell by an amount k_μ. The index i thus labels the off-shell incoming line with momentum $p_i - k$ and $p_i^2 = -m_i^2$, whereas j labels the outgoing line with momentum $p_j + k$ and $p_j^2 = -m_j^2$.

The reason why we consider the Born approximation diagrams of fig 11.2 separately, is that they become singular when the photon momentum k tends to zero because the propagator of the virtual particle diverges in that limit. Therefore we are entitled to restrict ourselves to the charge form factors $F_i(k^2)$ or $F_j(k^2)$ as they are measured for *real* particles, since the deviation from their on-shell value leads to terms in which the propagator pole cancels, and which are therefore regular when k tends to zero. Those terms are thus contained in the second part of (11.54), which is assumed to exhibit no singularities in the soft-photon limit.

We now impose current conservation on the full amplitude (11.54), i.e., we require that the amplitude vanishes when the photon polarization vector $\varepsilon_\mu(\mathbf{k})$ is replaced by k_μ. Contracting the momentum factors in the Born approximation amplitude with k_μ leads to the following factors

$$\begin{aligned} k_\mu \frac{(2p_i - k)^\mu}{(p_i - k)^2 + m_i^2} &= \frac{-(p_i - k)^2 + p_i^2}{(p_i - k)^2 + m_i^2} = -1\,, \\ k_\mu \frac{(2p_j + k)^\mu}{(p_j + k)^2 + m_j^2} &= \frac{(p_j + k)^2 - p_j^2}{(p_j + k)^2 + m_j^2} = +1\,. \end{aligned} \tag{11.56}$$

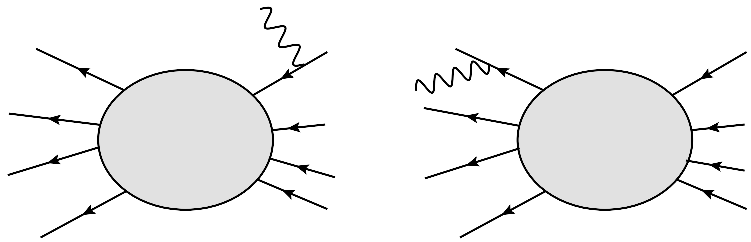

Figure 11.2: Born approximation diagrams corresponding to (11.55)

Using the fact that the remaining diagrams in (11.54) are regular for vanishing k, we thus find

$$-\sum_i \mathcal{M}(A[i] \to B)\, F_i(k^2) + \sum_j F_j(k^2)\, \mathcal{M}(A \to B[j]) = \mathrm{O}(k)\,, \tag{11.57}$$

or, in the soft-photon limit $k \to 0$,

$$\Big(\sum_j F_j(0) - \sum_i F_i(0)\Big)\, \mathcal{M}(A \to B) = 0\,, \tag{11.58}$$

where $\mathcal{M}(A \to B)$ is now the full on-shell amplitude for the process $A \to B$. The implication of (11.58) should be obvious: in every possible process the sum of the charges of the incoming particles should equal the sum of the charges of the outgoing particles. Consequently this result justifies the definition of electric charge as the value of the charge form factor at zero momentum transfer.

11.5. A low-energy theorem for pion-Compton scattering

In the previous section we have seen how the conservation of electric charge follows from the fact that photons must couple to conserved amplitudes. However, this feature has further consequences for the interactions with soft photons. We will illustrate this here for the case of Compton scattering on pions: $\gamma(k) + \pi^{\pm}(p) \to \gamma(k') + \pi^{\pm}(p')$. The amplitude for that process is split into a term coming from Born-approximation graphs and a remainder that is finite in the limit that the photon momenta vanish. Hence we write

$$\mathcal{M}_{\mu\nu}(p', k', p, k) \quad = \quad \mathcal{M}^{\mathrm{B}}_{\mu\nu}(p', k', p, k) + \mathcal{M}^{\mathrm{R}}_{\mu\nu}(p', k', p, k)\,, \tag{11.59}$$

where we have not yet contracted with the photon polarization vectors; the index μ refers to the outgoing photon, the index ν to the incoming one. The

Born graphs shown in fig 11.3 lead to the amplitude

$$\mathcal{M}^{\mathrm{B}}_{\mu\nu}(p',k',p,k) = F(k^2)\,F(k'^2) \times \left\{ \frac{(2p'+k')_\mu(2p+k)_\nu}{(p+k)^2+m^2} + \frac{(2p'-k)_\nu(2p-k')_\mu}{(p-k')^2+m^2} \right\}. \quad (11.60)$$

Contracting (11.60) with the photon momenta gives

$$\begin{aligned} k'^\mu \mathcal{M}^{\mathrm{B}}_{\mu\nu} &= 2F(k^2)\,F(k'^2)\,k'_\nu\,, \\ k^\nu \mathcal{M}^{\mathrm{B}}_{\mu\nu} &= 2F(k^2)F(k'^2)\,k_\mu\,, \end{aligned} \quad (11.61)$$

which shows that the amplitude $\mathcal{M}^{\mathrm{B}}$ does *not* satisfy current conservation. However, including an extra term proportional to $\eta_{\mu\nu}\,F(k^2)\,F(k'^2)$ in the Born approximation amplitude makes it separately conserved. The full amplitude thus decomposes as

$$\begin{aligned} \mathcal{M}_{\mu\nu}(p',k',p,k) &= F(k^2)\,F(k'^2) \\ &\times \left\{ \frac{(2p'+k')_\mu(2p+k)_\nu}{(p+k)^2+m^2} + \frac{(2p-k')_\mu(2p'-k)_\nu}{(p-k')^2+m^2} - 2\eta_{\mu\nu} \right\} \\ &+ \mathcal{M}^{\mathrm{R}}_{\mu\nu}(p',k',p,k)\,, \end{aligned} \quad (11.62)$$

so that the new $\mathcal{M}^{\mathrm{R}}$ should now also be separately conserved, i.e.,

$$k'^\mu \mathcal{M}^{\mathrm{R}}_{\mu\nu}(p',k',p,k) = k^\nu \mathcal{M}^{\mathrm{R}}_{\mu\nu}(p',k',p,k) = 0\,. \quad (11.63)$$

Note that the term proportional to $\eta_{\mu\nu}F(k^2)F(k'^2)$ is in accord with the presence of the $|\phi|^2 A_\mu^2$ interaction term in the Lagrangian for charged spinless fields cf. (11.34). How do we construct $\mathcal{M}^{\mathrm{R}}_{\mu\nu}$? One may begin by decomposing

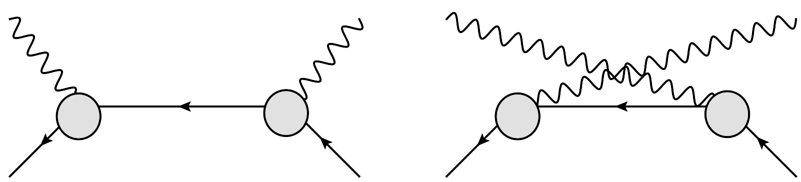

Figure 11.3: Born-approximation diagrams corresponding to (11.60).

the tensor into a basis formed by three linearly independent momenta and the tensor $\eta_{\mu\nu}$ (we ignore the tensor $\varepsilon_{\mu\nu\rho\sigma}$, because it violates parity conservation). Suppose we choose $P = \frac{1}{2}(p+p')$, k and k' as the linearly independent momenta. Then $\mathcal{M}^{\mathrm{R}}$ can be decomposed as

$$\begin{aligned} \mathcal{M}^{\mathrm{R}}_{\mu\nu} &= T_1\,\eta_{\mu\nu} + T_2\,k_\mu k_\nu + T_3\,k_\mu k'_\nu + T_4\,k_\mu P_\nu + T_5\,k'_\mu k_\nu \\ &+ T_6\,k'_\mu P_\nu + T_7\,k'_\mu k'_\nu + T_8\,P_\mu k_\nu + T_9\,P_\mu k'_\nu + T_{10}\,P_\mu P_\nu\,, \end{aligned} \quad (11.64)$$

where $T_1,\ldots,T_{10}$ are Lorentz-invariant functions. By imposing (11.63) one can easily verify that $T_1,\ldots,T_{10}$ can be expressed in terms of five independent functions. This is obvious by simply applying the transverse projection operators $(k'^2\eta_{\rho\mu} - k'_\rho k'_\mu)$ and $(k^2\eta_{\sigma\nu} - k_\sigma k_\nu)$ to $\mathcal{M}^{\rm R}_{\mu\nu}$, so that all terms in (11.64) proportional to k'_μ and k_ν cancel leaving five functions $T_1, T_3, T_4, T_9, T_{10}$. The corresponding tensors thus involve P_μ, P_ν, k'_μ, k_ν and $\eta_{\mu\nu}$, which after contraction with the transverse projection operators yields a representation for the conserved amplitude,

$$\begin{aligned}
\mathcal{M}^{\rm R}_{\mu\nu} \;=\;& \tilde T_1\,(k^2k'^2\,\eta_{\mu\nu} - k'^2\,k_\mu k_\nu - k^2\,k'_\mu k'_\nu + k\cdot k'\,k'_\mu k_\nu)\\
&+\tilde T_2\,(k'^2\,k_\mu - k\cdot k'\,k'_\mu)\,(k^2\,k'_\nu - k\cdot k'\,k_\nu)\\
&+\tilde T_3\,(k'^2\,k_\mu - k\cdot k'\,k'_\mu)\,(k^2\,P_\nu - P\cdot k\,k_\nu)\\
&+\tilde T_4\,(k'^2\,P_\mu - P\cdot k'\,k'_\mu)\,(k^2\,k'_\nu - k\cdot k'\,k_\nu)\\
&+\tilde T_5\,(k'^2\,P_\mu - P\cdot k'\,k'_\mu)\,(k^2\,P_\nu - P\cdot k\,k_\nu)\,,
\end{aligned}\tag{11.65}$$

where the precise relationship between the T's and the $\tilde T$'s can be found by comparing the two decompositions for the amplitude.

However, the disadvantage of (11.65) is that it is proportional to at least four powers of momenta k and k', while we only require that $\mathcal{M}$ is regular in the limits $k\to 0$ and $k'\to 0$. It is possible to set up an alternative decomposition that involves lower powers of k and k' by re-arranging the various terms and absorbing momentum factors into the invariant functions where possible. The result reads

$$\begin{aligned}
\mathcal{M}^{\rm R}_{\mu\nu} \;=\;& A\,\{k\cdot k'\,\eta_{\mu\nu} - k_\mu k'_\nu\}\\
&+B\,\{P\cdot k\,P\cdot k'\,\eta_{\mu\nu} - P\cdot k'\,k_\mu P_\nu - P\cdot k\,P_\mu k'_\nu + k\cdot k'\,P_\mu P_\nu\}\\
&+C\,\{k^2\,P\cdot k'\,\eta_{\mu\nu} - P\cdot k'\,k_\mu k_\nu - k^2\,P_\mu k'_\nu + k\cdot k'\,P_\mu k_\nu\}\\
&+D\,\{k'^2\,P\cdot k\,\eta_{\mu\nu} - P\cdot k\,k'_\mu k'_\nu - k'^2\,k_\mu P_\nu + k\cdot k'\,k'_\mu P_\nu\}\\
&+E\,\{k^2\,k'^2\,\eta_{\mu\nu} - k'^2\,k_\mu k_\nu - k^2\,k'_\mu k'_\nu + k\cdot k'\,k'_\mu k_\nu\}\,,
\end{aligned}\tag{11.66}$$

where A, B, C, D and E are the five linearly independent amplitudes we require. They are functions of k^2, k'^2, $P\cdot k$ and $k\cdot k'$ (or equivalently k^2, k'^2, $s=-(k+p)^2$, $t=-(k-k')^2$ and $u=-(k-p')^2$ with $s+t+u=2m^2-k^2-k'^2$). Actually there are further restrictions on these functions as a result of Bose symmetry, but they are not important for what follows. The above amplitude should remain the same under the simultaneous interchange of μ,ν and k,k'. However, for our purpose it is relevant that all terms in (11.66) are multiplied by factors that are at least quadratic in the photon momenta. The only way to further decrease these powers is by having some other propagator pole term contribute to A, B, C, D and E, but those would correspond to the pion-pole terms that we have already extracted. The full pion-Compton amplitude follows from substitution of (11.66) into (11.62). Contracting with (transverse) photon polarization vectors ε'_μ and ε_ν one obtains

$$\begin{aligned}\varepsilon'^{\mu}\,\varepsilon^{\nu}\,\mathcal{M}_{\mu\nu}(p',k',p,k) \;=\;& 2F^2(0)\Big\{\frac{\varepsilon'\cdot p'\,p\cdot\varepsilon}{p\cdot k}-\frac{\varepsilon'\cdot p\,p'\cdot\varepsilon}{p\cdot k'}-\varepsilon'\cdot\varepsilon\Big\}\\ &+A\Big\{k\cdot k'\;\varepsilon'\cdot\varepsilon-\varepsilon'\cdot k\;\varepsilon\cdot k'\Big\}\\ &+B\Big\{P\cdot k'\,(\varepsilon'\cdot\varepsilon\;P\cdot k-\varepsilon'\cdot k\;P\cdot\varepsilon)\\ &\quad-P\cdot\varepsilon'\,(k'\cdot\varepsilon\;P\cdot k-P\cdot\varepsilon\,k\cdot k')\Big\}\,,\end{aligned}\tag{11.67}$$

where we used that the photons are on their mass shell, $k^2 = k'^2 = 0$. Note that only two of the five unknown functions contribute. The interesting feature of (11.67) is that the first term in parentheses coincides precisely with the result obtained in the lowest-order of perturbation theory if we identify $F(0)$ with the pion charge, whereas all the additional terms are proportional to at least the square of the photon energy (we recall that in the laboratory frame the vanishing of the incoming photon energy ω implies the vanishing of the outgoing photon energy ω', and vice versa cf. (4.62), so that terms proportional to $\omega\omega', \omega^2$, and ω'^2 are all proportional to the square of the energy of either one of the photons). Therefore in the limit $\omega \to 0$ one obtains the same results as in section 4.4.

Let us work this out for the total cross section. Squaring the full amplitude and summing over photon polarizations yields

$$\begin{aligned}\tfrac{1}{2}\sum_{\rm pol}|\mathcal{M}|^2 \;=\;& 2e^4\Big\{m^4\Big(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\Big)^2-2m^2\Big(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\Big)+2\Big\}\\ &+e^2\Big(\frac{p'^{\,\mu}p^{\nu}}{p\cdot k}-\frac{p^{\mu}p'^{\,\nu}}{p\cdot k'}-\eta^{\mu\nu}\Big)\big(\mathcal{M}^{\rm R}_{\mu\nu}+\overline{\mathcal{M}}{}^{\rm R}_{\mu\nu}\big)\\ &+\mathcal{M}^{{\rm R}\mu\nu}\,\overline{\mathcal{M}}{}^{\rm R}_{\mu\nu}\,,\end{aligned}\tag{11.68}$$

where we substituted $F(0) = e$. From (11.67) we know that only the functions A and B will contribute to this result, as can be verified by explicit calculation (see problem 11.7).

Let us now consider this expression for small photon energies. In the laboratory frame one has $(p\cdot k)^{-1} - (p\cdot k')^{-1} = m^{-2}(1-\cos\theta)$, where θ is the photon deflection angle, so that the Born cross section does not contain a singularity in the soft-photon limit. The interference term of the Born amplitude with $\mathcal{M}^{\rm R}$ are formally of order ω, because $\mathcal{M}^{\rm R}$ is of order ω^2. However, decomposing the $1/\omega$ terms from the Born diagrams as

$$\frac{p'_\mu p_\nu}{p\cdot k} - \frac{p_\mu p'_\nu}{p\cdot k'} = \tfrac{1}{2}\left(p'_\mu p_\nu + p_\mu p'_\nu\right)\Big(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\Big) + \tfrac{1}{2}\left(p'_\mu p_\nu - p_\mu p'_\nu\right)\Big(\frac{1}{p\cdot k} + \frac{1}{p\cdot k'}\Big)\,, \tag{11.69}$$

shows that both these terms behave as ω^0, so that the total interference term is of order ω^2. Finally the last term proportional to the square of $\mathcal{M}^{\rm R}$ is of order ω^4. Using the lowest-order result (4.74), we thus obtain the following general result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\rm lab}} = \frac{\alpha^2}{2m^2}(1+\cos^2\theta)\Big\{1 - \frac{2\omega}{m}(1-\cos\theta) + \mathrm{O}(\omega^2)\Big\}. \tag{11.70}$$

This result shows that the electric charge, as defined in terms of the charge form factor at zero momentum transfer, is the same quantity that is measured in the Thomson scattering limit.

The above result was derived for spinless particles. However, it turns out to be more general because the spin-dependent corrections are always contained in the order ω^2 terms. One can see this, for instance, from the so-called Klein-Nishina formula, which gives the differential cross section for Compton scattering off electrons (derived in problem 7). It reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\rm lab}} = \frac{\alpha^2}{2m^2}\frac{1+\cos^2\theta}{\left[1+(\omega/m)\,(1-\cos\theta)\right]^2} \times\left[1 + \frac{\omega^2}{m^2}\frac{(1-\cos\theta)^2}{(1+\cos^2\theta)\left[1+(\omega/m)\,(1-\cos\theta)\right]}\right]. \tag{11.71}$$

The difference with the cross section (11.70) for spinless particles is of order $(\omega/m)^2$, so we see that the two results are consistent within the given approximation.

11.6. More photons

Now that we have gained some familiarity with the implications of gauge invariance for soft-photon processes, let us examine a slight extension of the theories discussed so far. We introduce an interacting field theory based on n gauge fields A^a_μ and N complex scalar fields ϕ^i of equal mass m (so that $a = 1,2,\ldots,n$ and $i = 1,2,\ldots,N$). The fields A^a_μ will describe massless particles so they should couple to conserved currents. Using the arguments of the preceding section one easily proves that the invariant amplitude for two on-shell scalar particles with a virtual vector particle can be parametrized as

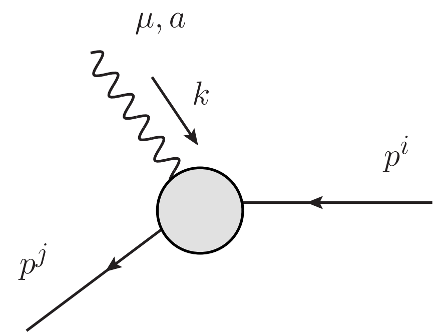

Figure 11.4: Diagram corresponding to (11.72).

cf. (11.50),(11.52)

$$\mathcal{M}^{ij}_{\mu a}(p^i, p^j) = F^{ij}_a(k^2)\,(p^i_\mu + p^j_\mu)\,, \tag{11.72}$$

where the momentum assignments are indicated in fig 11.4. The form factor $F^{ij}_a(k^2)$ is a direct generalization of the charge form factor introduced in the previous section.

Consider now the process in which an incoming scalar of type j and a photon of type b scatter and yield an outgoing scalar of type i and a photon of type a,

$$S^j(p^j) + \gamma^b_\nu(k^b) \longrightarrow S^i(p^i) + \gamma^a_\mu(k^a)\,. \tag{11.73}$$

For small photon momenta the amplitude may be approximated by the Born diagrams given in fig 11.5. The corresponding expression takes the form,

$$\begin{aligned}&\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i, k^a, p^j, k^b) = \\ &\quad F^{ik}_a(k^{a\,2})\,F^{kj}_b(k^{b\,2})\left\{\frac{(2p^i + k^a)_\mu\,(2p^j + k^b)_\nu}{(p^j + k^b)^2 + m^2} - \eta_{\mu\nu}\right\} \\ &\quad + F^{ik}_b(k^{b\,2})\,F^{kj}_a(k^{a\,2})\left\{\frac{(2p^i - k^b)_\nu\,(2p^j - k^a)_\mu}{(p^j - k^a)^2 + m^2} - \eta_{\mu\nu}\right\},\end{aligned} \tag{11.74}$$

where we have already included two regular terms proportional to $\eta_{\mu\nu}$ in direct analogy with a similar term in (11.62). Observe that we sum over the index k as we have to include all possible internal lines that can contribute to the diagram. Contracting (11.74) with photon momenta yields

$$
\begin{aligned}
k^{a\mu}\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i,k^a,p^j,k^b) &= \\
&\Big(F_a^{ik}(k^{a\,2})\,F_b^{kj}(k^{b\,2}) - F_b^{ik}(k^{b\,2})\,F_a^{kj}(k^{a\,2})\Big)\,(p^i+p^j)_\nu, \\
k^{b\nu}\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i,k^a,p^j,k^b) &= \\
&\Big(F_a^{ik}(k^{a\,2})\,F_b^{kj}(k^{b\,2}) - F_b^{ik}(k^{b\,2})\,F_a^{kj}(k^{a\,2})\Big)\,(p^i+p^j)_\mu,
\end{aligned} \tag{11.75}
$$

which only vanishes in the soft-photon limit $k^a, k^b \to 0$ if

$$
F_a^{ik}(0)\,F_b^{kj}(0) = F_b^{ik}(0)\,F_a^{kj}(0). \tag{11.76}
$$

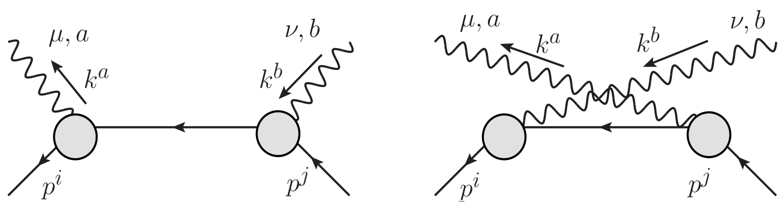

Figure 11.5: Born approximation diagrams corresponding to the singular part of (11.74).

As we have previously emphasized charges are defined by the zero-momentum value of their charge form factors. In this case we are thus dealing with n different charges, given by $F_a^{ij}(0)$, which take the form of $N\times N$ matrices. According to (11.76) these matrices must commute. Gauge theories with commuting charges are called *abelian*. For a *nonabelian* gauge theory not all charges commute, which seems to imply that the photons no longer couple to conserved currents. If this were indeed the case then the above argument would show the inconsistency of the nonabelian theories, so let us examine whether there could possibly be an additional contribution that becomes singular when one of the photon momenta tends to zero. Such a contribution must again correspond to a Born approximation graph with a photon attached to an external line. The only such graph that we have not yet considered corresponds to the one in which the two photons couple and interact with the scalars through the exchange of another photon. This diagram is shown in fig 11.6. As we shall see its contribution can indeed compensate for the undesired terms in (11.75) so that we will establish the important result that nonabelian gauge fields, i.e. gauge fields that couple to noncommuting charges, necessarily carry their own charge, thus leading to a self-interaction. This feature represents the distinctive difference between abelian and nonabelian theories.

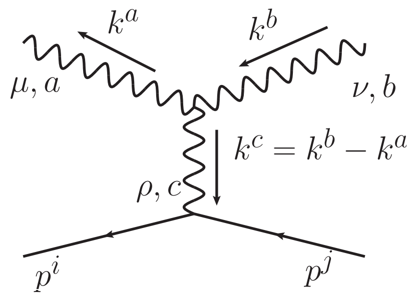

Figure 11.6: Additional Born approximation graph for nonabelian gauge fields.

To calculate the diagram in (11.6) we first determine the top vertex with three photons, with momenta chosen as in the figure. As we are mainly interested in soft photons we only write down the terms of first order in photon momenta. Because of Lorentz invariance these terms are linearly proportional to the external momenta. Requiring current conservation for on-shell momenta leads to

$$V_{\mu\nu\rho}(a,b,c) = \mathrm{i}F_{abc}\Big\{\eta_{\mu\nu}(l^a - l^b)_\rho + \eta_{\nu\rho}(l^b - l^c)_\mu + \eta_{\rho\mu}(l^c - l^a)_\nu\Big\} + \text{terms of higher order in momenta}\,, \qquad (11.77)$$

where F_{abc} is a constant and the momenta l^a, l^b and l^c are incoming here; the index assignments are indicated in fig 11.7. Because of Bose symmetry (11.77) must be symmetric under the simultaneous interchange of the momenta and indices associated with two external lines. This shows that F_{abc} must be antisymmetric in a, b, c. The constants F_{abc} represent the nonabelian charges of the photons themselves, in direct analogy with the charges $F_a^{ij}(0)$ for the scalar particles. To exhibit under precisely what conditions the vertex (11.77) is conserved let us contract it with one of the external momenta, say $l^{a\,\mu}$. This yields

$$l^{a\mu}V_{\mu\nu\rho}(a,b,c) = \mathrm{i}F_{abc}\Big\{[\eta_{\nu\rho}(l^c)^2 - l^c_\nu l^c_\rho] - [\eta_{\nu\rho}(l^b)^2 - l^b_\nu l^b_\rho]\Big\}\,, \qquad (11.78)$$

which vanishes when contracted with transverse polarization vectors $\varepsilon_\nu(\mathbf{l}^b)$, and $\varepsilon_\rho(\mathbf{l}^c)$, with $(l^b)^2 = (l^c)^2 = 0$, but not necessarily $(l^a)^2 = 0$. [2]

[2]Observe that Lorentz invariance also requires that $V_{\mu\nu\rho}$ vanishes on-shell upon contraction with $l^{a\mu}\, l^{b\nu}$ and $l^{a\mu}\, l^{b\nu}\, l^{c\rho}$; see problem 11.5.

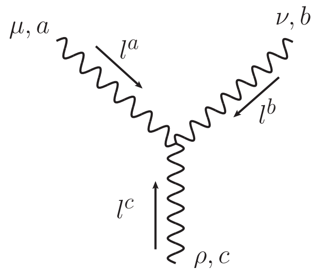

Figure 11.7: Three-point vertex for gauge fields.

It is now straightforward to evaluate the Born diagram of fig. 11.6. After replacing the unphysical momenta with those of the process in fig. 11.6, i.e. $l^a \to -k^a$, $l^b \to k^b$, $l^c \to -k^c$, we find

$$\begin{aligned}\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i,k^a,p^j,k^b) =& \\ &\mathrm{i}F_{abc}\Big\{-\eta_{\mu\nu}(k^a+k^b)_\rho+\eta_{\nu\rho}(2k^b-k^a)_\mu+\eta_{\rho\mu}(2k^a-k^b)_\nu\Big\} \\ &\times\left[\frac{\eta^{\rho\sigma}}{(p^i-p^j)^2}-(1-\lambda^{-2})\frac{(p^i-p^j)^\rho(p^i-p^j)^\sigma}{((p^i-p^j)^2)^2}\right] \\ &\times F^{ij}_c((p^i-p^j)^2)(p^i+p^j)_\sigma, \end{aligned} \tag{11.79}$$

where we have used the photon propagator (4.42) with an arbitrary gauge parameter λ. However the gauge dependent part of (11.77) cancels by virtue of $(p^i+p^j)\cdot(p^i-p^j)=0$, as the scalar particles are on the mass shell. Therefore we are left with

$$\begin{aligned}\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i,k^a,p^j,k^b) =& \frac{\mathrm{i}F_{abc}F^{ij}_c((p^i-p^j)^2)}{(p^i-p^j)^2} \\ &\times\big\{-\eta_{\mu\nu}(k^a+k^b)\cdot(p^i+p^j) \\ &\quad+(2k^b-k^a)_\mu(p^i+p^j)_\nu+(p^i+p^j)_\mu(2k^a-k^b)_\nu\big\}. \end{aligned} \tag{11.80}$$

Contracting this with $k^{a\mu}$ or $k^{b\nu}$ gives

$$\begin{aligned}k^{a\mu}\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i,k^a,p^j,k^b) =& -\mathrm{i}F_{abc}\,F^{ij}_c((p^i-p^j)^2) \\ &\times\left\{(p^i+p^j)_\nu-\frac{(k^b)^2\,(p^i+p^j)_\nu-k^a\cdot(p^i+p^j)\,k^b_\nu}{(p^i-p^j)^2}\right\}, \end{aligned} \tag{11.81}$$

and

$$\begin{aligned}k^{b\nu}\mathcal{M}^{\mathrm{B}ij}_{\mu\nu ab}(p^i,k^a,p^j,k^b) =& -\mathrm{i}F_{abc}\,F^{ij}_c((p^i-p^j)^2) \\ &\times\left\{(p^i+p^j)_\mu-\frac{(k^a)^2\,(p^i+p^j)_\mu-k^b\cdot(p^i+p^j)\,k^a_\mu}{(p^i-p^j)^2}\right\}, \end{aligned} \tag{11.82}$$

which should be compared to the corresponding results of the Born diagrams of fig 11.5, given in (11.75). As the second term in (11.81) and (11.82) vanishes when the photons are on the masss shell ($(k^a)^2 = (k^b)^2 = 0$) with transverse polarization (i.e., after contraction with the transverse photon polarization vectors $\varepsilon_\nu(\mathbf{k}^b)$ and $\varepsilon_\mu(\mathbf{k}^a)$), we restrict attention to the terms proportional to $p^i + p^j$.[3] Assuming the soft photon limit, $k^a, k^b \to 0$, we see that the combined result of (11.75), (11.81) and (11.82) then vanishes if

$$F_a^{ik}(0)F_b^{kj}(0) - F_b^{ik}(0)F_a^{kj}(0) = \mathrm{i}F_{abc}F_c^{ij}(0)\,, \tag{11.83}$$

so that we obtain a relation between the nonabelian charges of the scalar particles and the photons. This is the central result of this section, showing that if several gauge fields couple to noncommuting charges, here corresponding to the matrices $F_a^{ij}(0)$, then the corresponding massless vector particles carry charges F_{abc}, which characterize the lack of commutativity through (11.83). As this relation is nonlinear in the charges $F_a^{ij}(0)$, it is in general not possible to rescale these charges, a result we have been alluding to at the end of section 11.2. For the gauge field charges, F_{abc}, one expects a result similar to (11.83) with the $F_a^{ij}(0)$ replaced by F_{abc}. Indeed it turns out that

$$F_{adc}F_{bce} - F_{bdc}F_{ace} = -F_{abc}F_{cde}\,. \tag{11.84}$$

This result can be verified by repeating the calculation of this section for an amplitude with four external photons (see problem 11.6). Another way of seeing why (11.84) should hold is to consider the double commutator of the charges $F_a^{ij}(0)$ and use the Jacobi identity. Employing a matrix notation for the charges this gives

$$\begin{aligned}\big[[F_a(0), F_b(0)], F_d(0)\big] + \big[[F_b(0), F_d(0)], F_a(0)\big]&\\ +\big[[F_d(0), F_a(0)], F_b(0)\big] = 0\,.&\end{aligned} \tag{11.85}$$

Using (11.83) twice, one finds

$$-\left\{F_{abc}F_{cde} + F_{bdc}F_{cae} + F_{dac}F_{cbe}\right\}F_e^{ij}(0) = 0, \tag{11.86}$$

which implies (11.84) if one assumes that the charges $F_a^{ij}(0)$ are independent matrices. The above results are clearly indicative of some underlying mathematical structure. We have already seen that the way in which the gauge fields couple follows from the behaviour of the gauge transformations on the various fields. The presence of nonabelian charges suggests that these gauge transformations should define a *nonabelian* group, i.e., a group of which

[3]Note that for the abelian theory discussed in section 11.4, the amplitudes are conserved without the need for imposing a condition on the second photon. The significance of this fact will be discussed in section 13.3. Observe that the amplitude vanishes, however, upon contraction of both polarization vectors with the corresponding momenta.

not all elements commute. Indeed, (11.83) and (11.84) have a well-known group-theoretic meaning. We shall discuss this in the next chapter, where we introduce nonabelian gauge theories starting from nonabelian gauge transformations.

Problems

11.1. In this problem we examine how a transverse polarization vector transforms under Lorentz transformations. For simplicity consider a photon travelling along the z-axis with energy ω and polarization vector ε_μ. Apply the Lorentz boost given in (A.11) to both the momentum and polarization vectors to find the momentum $k'_\mu = L_\mu{}^\nu\, k_\nu$ and polarization vector $L_\mu{}^\nu\, \varepsilon_\nu$. As we have stated in the text (cf. 11.47), the polarization vector ε'_μ in the new frame can be expressed as follows,

$$L_\mu{}^\nu\, \varepsilon_\nu = \varepsilon'_\mu + \alpha\, k'_\mu\,. \tag{1}$$

Can you derive from general arguments why the relation (1) should hold for some value of α, using that ε'_μ is a polarization vector that has no fourth component? Subsequently, use this fact to show that for the Lorentz transformation (A.11) characterized by velocity components $\beta_1, \beta_2, \beta_3$, the parameter α is given by

$$\alpha = -\frac{\beta_i\, \varepsilon_i}{\omega(1-\beta_3)}\,. \tag{2}$$

Show that the three-vector components of k'_μ can be written as

$$k'_i = k_i - \frac{\omega\gamma}{1+\gamma}\big(1+\gamma(1-\beta_3)\big)\,\beta_i\,, \tag{3}$$

whereas ε'_i takes the form

$$\varepsilon'_i = \varepsilon_i - \alpha\, k_i + \frac{\alpha\omega\gamma}{1+\gamma}\,\beta_i\,. \tag{4}$$

Finally check that ε'_i is orthogonal to k'_i and that it has the same norm as the original polarization vector ε_i. Verify that polarization vectors orthogonal to the plane spanned by the three-momentum vector and the velocity vector of the Lorentz transformation remain unchanged.

11.2. Consider a covariant translation, defined as the combined variation induced by an infinitesimal displacement, $x^\mu \to x^\mu + a^\mu$ and a field-dependent gauge transformations with parameter $\xi = -a^\mu A_\mu$, but now acting on the gauge field. Show that the result equals $\delta A_\mu = a^\nu F_{\nu\mu}$. See also, problem 1.7.

11.3. In section 11.4 we showed how electric charge can be defined via the amplitude $\mathcal{M}_\mu(p',p)$ for a soft photon to couple to a (spinless) particle and demonstrated that current conservation implies that charge must be additively conserved in elementary particle reactions. Now we will do the analogous calculation for massless spin-2 particles. These could be gravitons and are described by a symmetric second-rank tensor field $g_{\mu\nu}$. For photons we have argued that the amplitude $\mathcal{M}_\mu(p',p)$ must satisy $(p'-p)^\mu \mathcal{M}_\mu(p',p) = 0$ in order to have Lorentz invariant interactions. For gravitons the amplitudes take the form of a symmetric tensor, so that the amplitude for massless graviton field $g^{\mu\nu}$ coupling to a spinless particle reads as $\mathcal{M}_{\mu\nu}(p',p)$. Because the gravitons are massless they must couple to a conserved amplitude to ensure Lorentz invariance, so that $(p'-p)^\mu \mathcal{M}_{\mu\nu}(p',p) = 0$. The covariant amplitude must be constructed from the four-vectors p' and p. It is convenient to write it in terms of $k_\mu = p'_\mu - p_\mu$ and $l_\mu = p'_\mu + p_\mu$. Show that the most general form allowed that is consistent with the above requirements is

$$\mathcal{M}_{\mu\nu}(p',p) = \frac{1}{2}\sqrt{G_\mathrm{N}}\big\{l_\mu l_\nu\, F(k^2) + (k^2\eta_{\mu\nu} - k_\mu k_\nu)H(k^2)\big\}\,, \tag{1}$$

where $F(k^2)$ and $H(k^2)$ are the (dimensionless) gravitational form factors, G_N is Newton's constant and we used $p^2 = p'^{\,2} = -m^2$.
Compare this result with the Fourier transform of the energy-momentum tensor given in (1.105) and (1.106); see also (3.15). In tree approximation both $F(k^2)$ and $H(k^2)$ are constant and equal to unity (unless we allow for an improvement term, in which case $H = \frac{2}{3}$). Note that (1) agrees with the general theory of relativity according to which the spin-2 graviton couples to the energy-momentum tensor.
Now consider the reaction $A \to B + g$ and work through the analogous steps for soft-graviton emission to show that

$$\Big(\sum_j F_j(0)\, p^j_\mu - \sum_i F_i(0)\, p^i_\mu\Big)\mathcal{M}(A \to B) = 0\,, \tag{2}$$

where i and j label the incoming and outgoing particles contained in the states A and B, respectively. Since this relation should hold for all particle reactions with arbitrary momentum configurations satisfying

$$\sum_i p^i_\mu = \sum_j p^j_\mu\,, \tag{3}$$

we must have all the $F_i(0)$ identical and equal to unity, so that soft gravitons couple with universal strength. This is in accordance with the principle of equivalence in general relativity.
Now analyse the scattering of two charged spinless particles with incoming momenta p and q and outgoing momenta p' and q'. Let their masses be M and m and their charges be e_1 and e_2, respectively. Suppose the scattering to take place via single photon and/or single graviton exchange (Born approximation). Define the usual invariants $s = -(p+q)^2$, $t = -(p-p')^2 = -(q-q')^2$, $u = -(p-q')^2$, satisfying $s+t+u = 2M^2 + 2m^2$ and four vectors $k = q'-q = p-p'$, $l = q'+q$, $r = p'+p$. To calculate the scattering amplitude we require the photon vertices proportional to $\mathrm{i}l_\mu$ and $\mathrm{i}r_\mu$ and the analogous graviton vertex (1). In the case of photon exchange we can use the propagator (4.5), while in the graviton case the following form can be used,

$$D_{\mu\nu\,\rho\sigma}(k^2) = \frac{1}{\mathrm{i}(2\pi)^4}\,\frac{1}{k^2}\big(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\big)\,. \tag{4}$$

Write down the Born amplitude for the one-photon and one-graviton exchange graphs and show that it can be written as

$$\mathcal{M} = \frac{1}{t}\Big\{e_1 e_2\,(s-u) + \frac{1}{4}G_{\rm N}\Big[F_1(t)\,F_2(t)\big(2(s-u)^2 - (t-4m^2)(t-4M^2)\big) -3t^2 H_1(t)\,H_2(t) + t(t-4m^2)F_1(t)\,H_2(t) + t(t-4M^2)H_1(t)\,F_2(t)\Big]\Big\}\,. \quad (5)$$

Consider the limit where the square of the momentum transfer $t \to 0$, and work in the rest frame where $\mathbf{p} = 0$ and $p_0 = M$. Write the momentum q_μ in terms of the energy E and the velocity $\vec{\beta}$ and show that

$$\mathcal{M} = -\frac{4E\,M}{t}\Big\{e_1 e_2 - G_{\rm N}\,E\,M(1+\beta^2)\Big\}\,. \quad (6)$$

When both particles are at rest, the first term corresponds to the Coulomb exchange force and the second term to the Newtonian gravitational exchange force, proportional to the product of the masses. Therefore the gravitational mass, defined via the exchange of soft gravitons, is equal to the inertial mass, an equality that is known as the equivalence principle. For massless particles the stituation is different as they have no mass and move at the speed of light. To see this consider also the case where one of the particles is massless ($m = 0$) and argue that this show that massless particles are affected by the gravitational force. This effect is very weak but shows up when massless particles pass heavy astronomical objects such as the sun. The above calculation does not quite apply to light rays, as we are only dealing with spinless particles, but similar expressions are obtained in the case of photons. Light bending of light rays by the sun has been observed experimentally is and this result constituted one of the early confirmations of Einstein's theory of gravity. For details consult S. Weinberg, Phys. Rev. 135 (1964) B1049, and M. Veltman, in: Methods in Field Theory (Les Houches Summer School No. 28) eds. R. Balian and J. Zinn-Justin (North Holland,1975) p.266. About the concept of mass in relativistic theories, see L.B. Okun, Physics Today, June 1989 and May 1990 issues.

11.4. One of the leptonic decay modes of the $\bar{\rm K}^0$ meson is $\bar{\rm K}^0(\mathbf{p}) \to \pi^+(\mathbf{q}) + e^-(\mathbf{q}_1) + \overline{\nu}_e(\mathbf{q}_2)$. This process is mediated by a W boson. The coupling of the W to the lepton pair follows from the Lagrangian (5.117). The K-π-W vertex for a virtual W of invariant mass $t = -(p-q)^2$ and an on-shell $\bar{\rm K}^0$ and π^+ is described by two phenomenological form factors $f_1(t)$ and $f_2(t)$:

$$\mathcal{M}_\mu(\bar{\rm K}^0\pi^+\mu W^-) = \frac{1}{4}\sqrt{2}\,g\sin\theta_C\,\big[f_1(t)(p+q)_\mu + f_2(t)(p-q)_\mu\big]\,, \quad (1)$$

where θ_C is the Cabibbo angle, discussed in chapter 5. In $\bar{\rm K}^0$ decay $m_e^2 \leqslant t \leqslant (m_{\rm K} - m_\pi)^2 \ll M_{\rm W}^2$ so that we can neglect the momentum dependence in the W-boson propagator. Hence the decay amplitude can be written as

$$\mathcal{M}(\bar{K}^0 \to \pi^+ e^- \overline{\nu}_e) = {\rm i}\frac{g^2}{8m_{\rm W}^2}\sin\theta_C\overline{u}(\mathbf{q_1}) \times \big[f_1(t)\,(\not p + \not q) + f_2(t)(\not p - \not q)\big]\,(1+\gamma_5)\,v(\mathbf{q_2})\,. \quad (2)$$

The shape of the form factors has been carefully measured for the allowed kinematical region of t (see references in the Review of Particle Properties, Phys. Lett. B239 (1990) 1).
In this problem we construct the amplitude for the electroweak reaction $\bar{\mathrm{K}}^0(\mathbf{p}) \to \pi^+(\mathbf{q}) + e^-(\mathbf{q_1}) + \overline{\nu}_e(\mathbf{q_2}) + \gamma(\mathbf{k})$, using the methods of 11.5. First use the electromagnetic coupling to pions and electrons to write the amplitudes for the diagrams where the photon is radiated from the outgoing pion and electron. This yields the result

$$\mathcal{M}^{\mathrm{B}}(\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e \gamma) = \varepsilon^\mu \mathcal{M}^{\mathrm{B}}_\mu, \tag{3}$$

where ε_μ is the polarization four-vector of the outgoing photon and

$$\begin{aligned} \mathcal{M}^{\mathrm{B}}_\mu \quad = \quad & \mathrm{i}\frac{e\,g^2\,\sin\theta_C}{8M_W^2} \\ & \times \overline{u}(\mathbf{q_1})\Big\{\frac{2q_\mu + k_\mu}{(q+k)^2 + m_\pi^2}\big[f_1(\tilde{t})\,(\not{p} + \not{q} + \not{k}) + f_2(\tilde{t})\,(\not{p} - \not{q} - \not{k})\big] \\ & -\gamma_\mu\,\frac{\not{q}_1 + \not{k} + i m_e}{(q_1+k)^2 + m_e^2}\big[f_1(t)(\not{p} + \not{q}) + f_2(t)(\not{p} - \not{q})\big]\Big\}(1+\gamma_5)\,v(\mathbf{q_2}), \end{aligned} \tag{4}$$

with $\tilde{t} = -(p-q-k)^2$. Now contract (4) with the photon momentum and use the Dirac equation to show that

$$\begin{aligned} k^\mu \mathcal{M}^{\mathrm{B}}_\mu \quad = \quad & \mathrm{i}\frac{e\,g^2\,\sin\theta_C}{8M_W^2}\,\overline{u}(\mathbf{q_1})\Big\{(f_1(\tilde{t}) - f_1(t))(\not{p} + \not{q}) \\ & +(f_2(\tilde{t}) - f_2(t))(\not{p} - \not{q}) + (f_1(\tilde{t}) - f_2(\tilde{t}))\not{k}\Big\}(1+\gamma_5)\,v(\mathbf{q_2})\,. \end{aligned} \tag{5}$$

Since (5) is nonzero we have to modify (4) by adding terms which are *regular* when $k \to 0$ in order to obtain a conserved amplitude (cf. 11.61). Because $\tilde{t} - t$ is proportional to k this is always possible. Show that the modified Born amplitude that satisfies current conservation is given by (we take $k^2 = 0$)

$$\begin{aligned} \mathcal{M}^{\mathrm{B}}_\mu \quad \to \quad & \mathcal{M}^{\mathrm{B}}_\mu - \mathrm{i}\frac{e\,g^2\,\sin\theta_C}{8M_W^2}\,\overline{u}(\mathbf{q_1})\Big\{(p-q)_\mu\big[\frac{f_1(\tilde{t}) - f_1(t)}{k\cdot(p-q)}(\not{p} + \not{q}) \\ & +\frac{f_2(\tilde{t}) - f_2(t)}{k\cdot(p-q)}(\not{p} - \not{q})\big] + (f_1(\tilde{t}) - f_2(\tilde{t}))\gamma_\mu\Big\}(1+\gamma_5)\,v(\mathbf{q_2})\,. \end{aligned} \tag{6}$$

Observe that the modification is indeed regular when $k \to 0$.
As discussed in section 11.5 the Born graphs describe the dominant contribution for the decay process when the photon momentum k tends to zero. Other contributions should be added to (6), but because (6) itself is conserved these additional contributions should be separately conserved. Show that the most general term that is conserved must be of the form

$$\begin{aligned} \mathcal{M}^{\mathrm{R}} \quad = \quad & \mathrm{i}\frac{e\,g^2\,\sin\theta_C}{8M_W^2}\,\overline{u}(\mathbf{q_1})\Big\{A(\varepsilon\not{k}\cdot p - \varepsilon\cdot p\,\not{k}) + B\varepsilon^{\mu\nu\rho\sigma}\varepsilon_\mu\gamma_\nu p_\rho k_\sigma \\ & +C(\varepsilon\not{k}\cdot q - \varepsilon\cdot q\,\not{k}) + D\varepsilon^{\mu\nu\rho\sigma}\varepsilon_\mu\gamma_\nu q_\rho k_\sigma + \mathrm{O}(k)\Big\}(1+\gamma_5)v(\mathbf{q_2})\,, \end{aligned} \tag{7}$$

where A, B, C and D are functions, regular for $k \to 0$. Because (7) is already of order k we may conclude that the result (6) must be correct up to that order. By expanding the form factors in a Taylor series show that (6) yields

$$\begin{aligned}\mathcal{M}^{\mathrm{B}}(\overline{K}^0 \to \pi^+ e^- \overline{\nu}_e \gamma) &= \mathrm{i}\frac{e\,g^2\,\sin\theta_C}{8M_W^2} \\ &\times \overline{u}(\mathbf{q_1})\Big\{\Big(\frac{\varepsilon\cdot q}{k\cdot q} - \frac{\varepsilon\cdot q_1}{k\cdot q_1} - \frac{\not\varepsilon\not k}{2k\cdot q_1}\Big)\big[2f_1(t)\not p - im_e(f_1(t) - f_2(t))\big] \\ &\quad -2\Big(\varepsilon\cdot p - \frac{\varepsilon\cdot q}{k\cdot q}k\cdot p\Big)\big(2f_1'(t)\not p - im_e(f_1'(t) - f_1'(t))\big) + \mathrm{O}(k)\Big\}(1+\gamma_5)v(\mathbf{q_2}),\end{aligned} \tag{8}$$

where $f_1' = \mathrm{d}f_1/\mathrm{d}t$ and the terms in m_e follow from the use of the Dirac equation. Would the inclusion of an electromagnetic form factor for the π^+ modify this result? The experiment by K.J. Peach et al, Phys. Lett. 35B (1971) 351 detected the higher-order terms in k. A theoretical reference is H.W. Fearing, E. Fischbach and J. Smith, Phys. Rev. D2 (1970) 542.

11.5. In subsection 11.6 we introduced several inequivalent 'photons' with mutual interactions resulting in a three-photon vertex (11.77) valid to terms of first order in the momenta. Contract (11.77) with the momentum $k^{a\,\mu}$ and derive the result (11.78). Multiply (11.78) with the momentum vector $k^{b\,\nu}$ and show that

$$k^{a\,\mu}\,k^{b\,\nu}\,V_{\mu\nu\rho}(k^a,k^b,k^c) = \tfrac{1}{2}\mathrm{i}F_{abc}\big\{(k^c)^2\,(k^b_\rho - k^a_\rho) + [(k^b)^2 - (k^a)^2]\,k^c_\rho\big\}\,, \tag{1}$$

which, after contraction with the polarization vector $\varepsilon^c_\rho \equiv \varepsilon_\rho(\mathbf{k^c})$, vanishes when $(k^c)^2 = 0$. Finally show that

$$k^{a\,\mu}\,k^{b\,\nu}\,k^{c\,\rho}\,V_{\mu\nu\rho}(k^a,k^b,k^c) = 0. \tag{2}$$

Argue why these results are necessary for Lorentz invariance.

11.6. We will use (11.77) and the photon propagator (4.42) to construct the amplitude for the scattering of inequivalent soft 'photons' labelled by indices μ,ν,ρ,σ and incoming momenta k^a,k^b,k^c,k^d, so that $k^a+k^b+k^c+k^d=0$. In tree approximation there are three diagrams which contribute to this amplitude, where a virtual 'photon' with label e is exchanged. Denote the amplitude for the channel $a+b\to e\to c+d$, by $\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma} \equiv \mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma}(k^a,k^b,k^c,k^d)$. Show that the other amplitudes are related by the exchange of indices and momenta so that

$$\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma}(\mathrm{total}) = \mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma} + \mathcal{M}^{a\,c\,d\,b}_{\mu\rho\sigma\nu} + \mathcal{M}^{a\,d\,b\,c}_{\mu\sigma\nu\rho}. \tag{1}$$

Derive that

$$\begin{aligned}\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma} \;=\;& \mathrm{i}F_{abe}\Big[(-2k^a - k^b)_\nu\eta_{\mu\tau} + (k^a+2k^b)_\mu\eta_{\nu\tau} + (k^a-k^b)_\tau\eta_{\mu\nu}\Big] \\ &\times\frac{1}{(k^a+k^b)^2}\Big[\eta_{\tau\alpha} + (1-\lambda^{-2})\frac{(k^a+k^b)_\tau(k^c+k^d)_\alpha}{(k^a+k^b)^2}\Big] \\ &\times\mathrm{i}F_{ecd}\Big[(-2k^c-k^d)_\sigma\eta_{\alpha\rho} + (k^c-k^d)_\alpha\eta_{\rho\sigma} + (k^c+2k^d)_\rho\eta_{\sigma\alpha}\Big].\end{aligned} \tag{2}$$

Multiply (2) with transverse polarization vectors $\varepsilon^a_\mu \equiv \varepsilon_\mu(\mathbf{k}^a), \varepsilon^b_\nu \equiv \varepsilon_\nu(\mathbf{k}^b), \varepsilon^c_\rho \equiv \varepsilon_\rho(\mathbf{k}^c), \varepsilon^d_\sigma \equiv \varepsilon_\sigma(\mathbf{k}^d)$, and include the vertex (11.77) to show that $\varepsilon^{a\mu}\varepsilon^{b\nu}\varepsilon^{c\rho}\varepsilon^{d\sigma}\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma}$ is independent of the gauge parameter for $(k^a)^2 = (k^b)^2 = (k^c)^2 = (k^d)^2 = 0$.
Contract (2) with $k^{a\,\mu}$, $\varepsilon^{b\nu}$, $\varepsilon^{c\rho}$, and ε^d_σ, and use the results of the problem 11.5 to show that for $(k^b)^2 = (k^c)^2 = (k^d)^2 = 0$,

$$\begin{aligned} k^{a\mu}\varepsilon^{b\,\nu}\varepsilon^{c\,\rho}\varepsilon^{d\,\rho}\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma} \;=\;& F_{aeb}F_{ecd}\big[\varepsilon^b\!\cdot\!\varepsilon^c\,\varepsilon^d\!\cdot\!(k^b-k^c) + \varepsilon^c\!\cdot\!\varepsilon^d\,\varepsilon^b\!\cdot\!(k^c-k^d) \\ & +\varepsilon^d\!\cdot\!\varepsilon^b\,\varepsilon^c\!\cdot\!(k^d-k^b) + k^{a\,\mu}\,\varepsilon^{b\,\nu}\,\varepsilon^{c\,\rho}\,\varepsilon^{d\,\sigma}\,t_{\mu\nu\rho\sigma}\big]. \end{aligned} \quad (3)$$

Give the explicit form of the constant tensor $t_{\mu\nu\rho\sigma}$.
Consider first the limit that $k^a \to 0$, and add the contributions from all three diagrams. Show that $k^{a\,\mu}\,\varepsilon^{b\nu}\,\varepsilon^{c\,\rho}\,\varepsilon^{d\,\sigma}\,\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma}(\text{total})$ vanishes if

$$F_{abe}\,F_{edc} + F_{bde}\,F_{eac} + F_{dae}\,F_{ebc} = 0. \quad (4)$$

Observe that this is the result alluded to in section 11.6 (cf.11.79).
Now reconsider the result (3) for $k^a_\mu \neq 0$. Since $t_{\mu\nu\rho\sigma}$ is multiplied with the same momentum k^a_μ as the left hand side of (3) we can account for this term by making a modification to $\mathcal{M}$ according to

$$\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma} \to \mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma} - F_{aeb}\,F_{ecd}\,t_{\mu\nu\rho\sigma}. \quad (5)$$

Making corresponding replacements for the other diagrams it immediately follows that

$$k^{a\,\mu}\,\varepsilon^{b\,\nu}\,\varepsilon^{c\,\rho}\,\varepsilon^{d\,\sigma}\,\mathcal{M}^{a\,b\,c\,d}_{\mu\nu\rho\sigma}(\text{total}) = 0. \quad (6)$$

Argue that the modification in (5) corresponds to adding a new momentum-independent primary vertex for the interactions between four 'photons'

$$V_{\mu\nu\rho\sigma} = -F_{abe}\,F_{edc}\,t_{\mu\nu\rho\sigma} - F_{bde}\,F_{eac}\,t_{\mu\rho\sigma\nu} - F_{dae}\,F_{ebc}\,t_{\mu\sigma\nu\rho} \quad (7)$$

This demonstrates that a nonabelian gauge theory of massless gauge fields requires three-field and four-field interaction terms in the Lagrangian.

11.7. Consider the case of Compton scattering of photons by electrons. To derive the scattering amplitude we can use the results from (9.120)-(9.126) for the decay of a virtual photon $\gamma^*(\mathbf{Q}) \to \mu^+(\mathbf{p}) + \mu^-(\mathbf{p}') + \gamma(\mathbf{k})$ by implementing crossing to switch the final outgoing antifermion into an incoming fermion. Rewrite the Born amplitudes corresponding to those in (9.120) for the scattering reaction $\gamma(\mathbf{Q}) + \mathrm{e}^-(\mathbf{p}) \to \gamma(\mathbf{k}) + \mathrm{e}^-(\mathbf{p}')$ and show that the square of the amplitude summed over all electron spins can be obtained from (9.121) by the substitution $p_\mu \to -p_\mu$ together with an overall change of sign (because we replace v spinors by u spinors in the sum (cf. 5.72-73)). Therefore the square of the amplitude summed over photon polarizations and fermion spins is obtainable from (9.126), i.e.,

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}|^2 \;=\;& 16e^4 \Big\{ 2(p\!\cdot\!k' + 2m^2)\Big(\frac{-2p\!\cdot\!k'}{NN'} + \frac{m^2}{N^2} + \frac{m^2}{N'^2}\Big) \\ & +(2p\!\cdot\!p' - m^2)\Big(\frac{1}{N} + \frac{1}{N'}\Big) \\ & -\frac{1}{2}\Big(\frac{N}{N'} + \frac{N'}{N}\Big) - m^2\Big(\frac{N'}{N^2} + \frac{N}{N'^2}\Big)\Big\}, \end{aligned} \quad (1)$$

where $N = -2k\cdot p$, $N' = 2k\cdot p'$. Now use $2p\cdot p' = N + N' - 2m^2$ to derive

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \Big\{ -\frac{N}{N'} - \frac{N'}{N} - 4m^2\Big(\frac{1}{N} + \frac{1}{N'}\Big) + 4m^4\Big(\frac{1}{N} + \frac{1}{N'}\Big)^2 \Big\}, \qquad (2)$$

where the factor $\frac{1}{4}$ was added because we want to average over the spins of the incoming photon and electron. In the laboratory frame (cf. 4.59) where where the electron is at rest, the incoming photon energy equals ω and the outgoing photon has energy ω' and is deflected over an angel θ, show that $N = 2m\omega'$, $N' = -2m\omega$, where $\omega^{-1} - \omega'^{-1} = -m^{-1}(1 - \cos\theta)$ (cf. 4.60). Then using (4.70) and (4.71), derive the angular distribution of the outgoing photon which is given by the Klein-Nishina formula (11.71). The higher-order contribution to the amplitudes constructed according to the prescription in section 11.5 do not contribute to the $\mathrm{O}(\omega/m)$ term in the expansion of (11.71). Ignoring all terms of order $(\omega/m)^2$, we see that we obtain the universal result (11.70) in the Thomson scattering limit.