1
Gauge Invariance

1.1
Introduction

Gauge field theories have revolutionized our understanding of elementary particle interactions during the second half of the twentieth century. There is now in place a satisfactory theory of strong and electroweak interactions of quarks and leptons at energies accessible to particle accelerators at least prior to LHC.

All research in particle phenomenology must build on this framework. The purpose of this book is to help any aspiring physicist acquire the knowledge necessary to explore extensions of the standard model and make predictions motivated by shortcomings of the theory, such as the large number of arbitrary parameters, and testable by future experiments.

Here we introduce some of the basic ideas of gauge field theories, as a starting point for later discussions. After outlining the relationship between symmetries of the Lagrangian and conservation laws, we first introduce global gauge symmetries and then local gauge symmetries. In particular, the general method of extending global to local gauge invariance is explained.

For global gauge invariance, spontaneous symmetry breaking gives rise to massless scalar Nambu–Goldstone bosons. With local gauge invariance, these unwanted particles are avoided, and some or all of the gauge particles acquire mass. The simplest way of inducing spontaneous breakdown is to introduce scalar Higgs fields by hand into the Lagrangian.

1.2
Symmetries and Conservation Laws

A quantum field theory is conveniently expressed in a Lagrangian formulation. The Lagrangian, unlike the Hamiltonian, is a Lorentz scalar. Further, important conservation laws follow easily from the symmetries of the Lagrangian density, through the use of Noether's theorem, which is our first topic. (An account of Noether's theorem can be found in textbooks on quantum field theory, e.g., Refs. [1] and [2].)

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Later we shall become aware of certain subtleties concerning the straightforward treatment given here. We begin with a Lagrangian density

$$\mathscr{L}\big(\phi_k(x), \partial_\mu \phi_k(x)\big) \tag{1.1}$$

where $\phi_k(x)$ represents genetically all the local fields in the theory that may be of arbitrary spin. The Lagrangian $L(t)$ and the action S are given, respectively, by

$$L(t) = \int d^3x \mathscr{L}\big(\phi_k(x), \partial_k \phi_k(x)\big) \tag{1.2}$$

and

$$s \int_{t_1}^{t_2} dt L(t) \tag{1.3}$$

The equations of motion follow from the Hamiltonian principle of stationary action,

$$\delta S = \delta \int_{t_1}^{t_2} dt\, d^3x \mathscr{L}\big(\phi_k(x), \partial_\mu \phi_k(x)\big) \tag{1.4}$$

$$= 0 \tag{1.5}$$

where the field variations vanish at times t_1 and t_2 which may themselves be chosen arbitrarily.

It follows that (with repeated indices summed)

$$0 = \int_{t_1}^{t_2} dt\, d^3x \left[\frac{\partial \mathscr{L}}{\partial \phi_k} \delta\phi_k + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)} \delta(\partial_\mu \phi_k) \right] \tag{1.6}$$

$$= \int_{t_1}^{t_2} dt\, d^3x \left[\frac{\partial \mathscr{L}}{\partial \phi_k} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)} \right] \delta\phi_k + \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)} \delta\phi_k \right]_{t=t_1}^{t=t_2} \tag{1.7}$$

and hence

$$\frac{\partial \mathscr{L}}{\partial \phi_k} = \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)} \tag{1.8}$$

which are the Euler–Lagrange equations of motion. These equations are Lorentz invariant if and only if the Lagrangian density $\mathscr{L}$ is a Lorentz scalar.

The statement of Noether's theorem is that to every continuous symmetry of the Lagrangian there corresponds a conservation law. Before discussing internal symmetries we recall the treatment of symmetry under translations and rotations.

Since $\mathscr{L}$ has no explicit dependence on the space–time coordinate [only an implicit dependence through $\phi_k(x)$], it follows that there is invariance under the translation

$$x_\mu \to x'_\mu = x_\mu + a_\mu \tag{1.9}$$

where a_μ is a four-vector. The corresponding variations in $\mathscr{L}$ and $\phi_k(x)$ are

$$\delta\mathscr{L} = a_\mu \partial_\mu \mathscr{L} \tag{1.10}$$

$$\delta\phi_k(x) = a_\mu \partial_\mu \phi_k(x) \tag{1.11}$$

Using the equations of motion, one finds that

$$a_\mu \partial_\mu \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi_k}\delta\phi_k + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)}\delta(\partial_\mu \phi_k) \tag{1.12}$$

$$= \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)}\delta\phi_k \right] \tag{1.13}$$

$$= a_\nu \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)}\partial_\nu \phi_k \right] \tag{1.14}$$

If we define the tensor

$$T_{\mu\nu} = -g_{\mu\nu}\mathscr{L} + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_k)}\partial_\nu \phi_k \tag{1.15}$$

it follows that

$$\partial_\mu T_{\mu\nu} = 0 \tag{1.16}$$

This enables us to identify the four-momentum density as

$$\mathscr{P}_\mu = T_{0\mu} \tag{1.17}$$

The integrated quantity is given by

$$P_\mu = \int d^3x\, \mathscr{P}_\mu \tag{1.18}$$

$$= \int d^3x (-g_{0\mu}\mathscr{L} + \pi_k \partial_\mu \phi_k) \tag{1.19}$$

where $\pi_k = \partial\mathscr{L}/\partial\dot{\phi}_k$ is the momentum conjugate to ϕ_k. Notice that the time component is

$$\mathscr{P}_0 = \pi_k \partial_0 \phi_k - \mathscr{L} \tag{1.20}$$

$$= \mathscr{H} \tag{1.21}$$

where $\mathscr{H}$ is the Hamiltonian density. Conservation of linear momentum follows since

$$\frac{\partial}{\partial t} P_\mu = 0 \tag{1.22}$$

This follows from $P_i = J_{0i}$ and $\frac{\partial}{\partial t} J_{0i}$ becomes a divergence that vanishes after integration $\int d^3x$.

Next we consider an infinitesimal Lorentz transformation

$$x_\mu \rightarrow x'_\mu = x_\mu + \epsilon_{\mu\nu} x_\nu \tag{1.23}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. Under this transformation the fields that may have nonzero spin will transform as

$$\phi_k(x) \rightarrow \left(\delta_{kl} - \frac{1}{2}\epsilon_{\mu\nu}\Sigma_{kl}^{\mu\nu}\right)\phi_l(x') \tag{1.24}$$

Here $\Sigma_{kl}^{\mu\nu}$ is the spin transformation matrix, which is zero for a scalar field. The factor $\frac{1}{2}$ simplifies the final form of the spin angular momentum density.

The variation in $\mathscr{L}$ is, for this case,

$$\delta\mathscr{L} = \epsilon_{\mu\nu} x_\nu \partial_\mu \mathscr{L} \tag{1.25}$$

$$= \partial_\mu(\epsilon_{\mu\nu} x_\nu \mathscr{L}) \tag{1.26}$$

since $\epsilon_{\mu\nu}\partial_\mu x_\nu = \epsilon_{\mu\nu}\delta_{\mu\nu} = 0$ by antisymmetry.

We know, however, from an earlier result that

$$\delta\mathscr{L} = \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}\delta\phi_k\right] \tag{1.27}$$

$$= \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}\left(\epsilon_{\lambda\nu}x_\nu\partial_\lambda\phi_k - \frac{1}{2}\Sigma_{kl}^{\lambda\nu}\epsilon_{\lambda\nu}\phi_l\right)\right] \tag{1.28}$$

It follows by subtracting the two expressions for $\delta\mathscr{L}$ that if we define

$$\mathscr{M}^{\lambda\nu\nu} = \left(x^\nu g^{\lambda\mu} - x^\mu g^{\lambda\nu}\right)\mathscr{L} + \frac{\partial}{\partial(\partial_\lambda\phi_k)}\left[\left(x^\mu\partial_\nu - x^\nu\partial_\mu\right)\phi_k + \Sigma_{kl}^{\mu\nu}\phi_l\right] \tag{1.29}$$

$$= x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + \frac{\partial\mathscr{L}}{\partial(\partial_\lambda\pi_k)}\Sigma_{kl}^{\mu\nu}\phi_l \tag{1.30}$$

then

$$\partial_\lambda \mathscr{M}^{\lambda\mu\nu} = 0 \tag{1.31}$$

The Lorentz generator densities may be identified as

$$\mathscr{M}^{\mu\nu} = \mathscr{M}^{0\mu\nu} \tag{1.32}$$

Their space integrals are

$$M^{\mu\nu} = \int d^3x \mathscr{M}^{\mu\nu} \tag{1.33}$$

$$= \int d^3x \left(x^\mu \mathscr{P}^\nu - x^\nu \mathscr{P}^\mu + \pi_k \Sigma_{kl}^{\mu\nu} \phi_l\right) \tag{1.34}$$

and satisfy

$$\frac{\partial}{\partial t} M^{\mu\nu} = 0 \tag{1.35}$$

The components M^{ij} $(i, j = 1, 2, 3)$ are the generators of rotations and yield conservation of angular momentum. It can be seen from the expression above that the contribution from orbital angular momentum adds to a spin angular momentum part involving $\Sigma_{kl}^{\mu\nu}$.

The components M^{0i} generate boosts, and the associated conservation law [3] tells us that for a field confined within a finite region of space, the "average" or center of mass coordinate moves with the uniform velocity appropriate to the result of the boost transformation (see, in particular, Hill [4]). This then completes the construction of the 10 Poincaré group generators from the Lagrangian density by use of Noether's theorem.

Now we may consider internal symmetries, that is, symmetries that are not related to space–time transformations. The first topic is global gauge invariance; in Section 1.3 we consider the generalization to local gauge invariance.

The simplest example is perhaps provided by electric charge conservation. Let the finite gauge transformation be

$$\phi_k(x) \to \phi_k'(x) = e^{-iq_k} \phi_k(x) \tag{1.36}$$

where q_k is the electric charge associated with the field $\phi_k(x)$. Then every term in the Lagrangian density will contain a certain number m of terms

$$\phi_{k_1}(x)\phi_{k_2}(x) \cdots \phi_{k_m}(x) \tag{1.37}$$

which is such that

$$\sum_{i=1}^{m} q_{k_i} = 0 \tag{1.38}$$

and hence is invariant under the gauge transformation. Thus the invariance implies that the Lagrangian is electrically neutral and all interactions conserve electric charge. The symmetry group is that of unitary transformations in one dimension, U(1). Quantum electrodynamics possesses this invariance: The uncharged photon has $q_k = 0$, while the electron field and its conjugate transform, respectively, according to

$$\psi \to e^{-iq\theta}\psi \tag{1.39}$$

$$\bar{\psi} \to e^{+iq\theta}\bar{\psi} \tag{1.40}$$

where q is the electronic charge.

The infinitesimal form of a global gauge transformation is

$$\phi_k(x) \to \phi_k(x) - i\epsilon^i \lambda^i_{kl}\phi_l(x) \tag{1.41}$$

where we have allowed a nontrivial matrix group generated by λ^i_{kl}. Applying Noether's theorem, one then observes that

$$\delta\mathscr{L} = \partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}\delta\phi_k\right] \tag{1.42}$$

$$= -i\epsilon^i\partial_\mu\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}\lambda^i_{kl}\phi_l\right] \tag{1.43}$$

The currents conserved are therefore

$$J^i_\mu = -i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}\lambda^i_{kl}\phi_l \tag{1.44}$$

and the charges conserved are

$$Q^i = \int d^3x j^i_0 \tag{1.45}$$

$$= -i\int d^3x\pi_k\lambda^i_{kl}\phi_l \tag{1.46}$$

satisfying

$$\frac{\partial}{\partial t}Q^i = 0 \tag{1.47}$$

The global gauge group has infinitesimal generators Q_i; in the simplest case, as in quantum electrodynamics, where the gauge group is U(1), there is only one such generator Q of which the electric charges q_k are the eigenvalues.

1.3
Local Gauge Invariance

In common usage, the term *gauge field theory* refers to a field theory that possesses a local gauge invariance. The simplest example is provided by quantum electrodynamics, where the Lagrangian is

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{1.48}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu \tag{1.49}$$

Here the slash notation denotes contraction with a Dirac gamma matrix: $A\!\!\!/ \equiv \gamma_\mu A_\mu$. The Lagrangian may also be written

$$\mathscr{L} = \bar{\psi}(iD\!\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{1.50}$$

where $D_\mu\psi$ is the covariant derivative (this terminology will be explained shortly)

$$D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi \tag{1.51}$$

The global gauge invariance of quantum electrodynamics follows from the fact that $\mathscr{L}$ is invariant under the replacement

$$\psi \to \psi' = e^{i\theta}\psi \tag{1.52}$$

$$\bar{\psi} \to \bar{\psi}' = e^{-i\theta}\bar{\psi} \tag{1.53}$$

where θ is a constant; this implies electric charge conservation. Note that the photon field, being electrically neutral, remains unchanged here.

The crucial point is that the Lagrangian $\mathscr{L}$ is invariant under a much larger group of local gauge transformations, given by

$$\psi \to \psi' = e^{i\theta(x)}\psi \tag{1.54}$$

$$\bar{\psi} \to \bar{\psi}' = e^{-i\theta(x)}\bar{\psi} \tag{1.55}$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta(x) \tag{1.56}$$

Here the gauge function $\theta(x)$ is an arbitrary function of x. Under the transformation, $F_{\mu\nu}$ is invariant, and it is easy to check that

$$\bar{\psi}'(i\partial\!\!\!/ - eA\!\!\!/')\psi' = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi \tag{1.57}$$

so that $\bar{\psi} \not{D} \psi$ is invariant also. Note that the presence of the photon field is essential since the derivative is invariant only because of the compensating transformation of A_μ. By contrast, in global transformations where θ is constant, the derivative terms are not problematic.

Note that the introduction of a photon mass term $-m^2 A_\mu A_\mu$ into the Lagrangian would lead to a violation of local gauge invariance; in this sense we may say that physically the local gauge invariance corresponds to the fact that the photon is precisely massless.

It is important to realize, however, that the requirement of local gauge invariance does not imply the existence of the spin-1 photon, since we may equally well introduce a derivative

$$A_\mu = \partial_\mu \Lambda \tag{1.58}$$

where the scalar Λ transforms according to

$$\Lambda \to \Lambda' = \Lambda - \frac{1}{e}\theta \tag{1.59}$$

Thus to arrive at the correct $\mathscr{L}$ for quantum electrodynamics, an additional assumption, such as renormalizability, is necessary.

The local gauge group in quantum electrodynamics is a trivial Abelian U(1) group. In a classic paper, Yang and Mills [5] demonstrated how to construct a field theory locally invariant under a non-Abelian gauge group, and that is our next topic.

Let the transformation of the fields $\phi_k(x)$ be given by

$$\delta\phi_k(x) = -i\theta^i(x)\lambda^i_{kl}\phi_l(x) \tag{1.60}$$

so that

$$\phi_k(x) \to \phi'_k(x) = \Omega_{kl}\phi_l \tag{1.61}$$

with

$$\Omega_{kl} = \delta_{kl} - i\theta^i(x)\lambda^i_{kl} \tag{1.62}$$

where the constant matrices λ^i_{kl} satisfy a Lie algebra $(i, j, k = 1, 2, \ldots, n)$

$$\left[\lambda^i, \lambda^j\right] = i c_{ijk}\lambda^k \tag{1.63}$$

and where the $\theta^i(x)$ are arbitrary functions of x.

Since Ω depends on x, a derivative transforms as

$$\partial_\mu\phi_k \to \Omega_{kl}(\partial_\mu\phi_l) + (\partial_\mu\Omega_{kl})\phi_l \tag{1.64}$$

We now wish to construct a *covariant* derivative $D_\mu\phi_k$ that transforms according to

$$D_\mu\phi_k \to \Omega_{kl}(D_\mu\phi_l) \tag{1.65}$$

To this end we introduce n gauge fields A^i_μ and write

$$D_\mu \phi_k = (\partial_\mu - igA_\mu)\phi_k \tag{1.66}$$

where

$$A_\mu = A^i_\mu \lambda^i \tag{1.67}$$

The required transformation property follows provided that

$$(\partial_\mu \Omega)\phi - igA'_\mu \Omega\phi = -ig(\Omega A_\mu)\phi \tag{1.68}$$

Thus the gauge field must transform according to

$$A_\mu \to A'_\mu = \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1} \tag{1.69}$$

Before discussing the kinetic term for A^i_μ it is useful to find explicitly the infinitesimal transformation. Using

$$\Omega_{kl} = \delta_{kl} - i\lambda^i_{kl}\theta^i \tag{1.70}$$

$$\Omega^{-1}_{kl} = \delta_{kl} + i\lambda^i_{kl}\theta^i \tag{1.71}$$

one finds that

$$\lambda^i_{kl}A'^i_\mu = \Omega_{km}\lambda^i_{mn}A^i_\mu\left(\Omega^{-1}\right) - \frac{i}{g}(\partial_\mu \Omega_{km})\left(\Omega^{-1}\right)_{ml} \tag{1.72}$$

so that (for small θ^i)

$$\lambda^i_{kl}\delta A^i_\mu = i\theta^j\left[\lambda^i, \lambda^j\right]_{kl}A^i_\mu - \frac{1}{g}\lambda^i_{kl}\partial_\mu\theta^i \tag{1.73}$$

$$= -\frac{1}{g}\lambda^i_{kl}\partial_\mu\theta^i - c_{ijm}\theta^j A^i_\mu \lambda^m_{kl} \tag{1.74}$$

This implies that

$$\delta A^i_\mu = -\frac{1}{g}\partial_\mu\theta^i + c_{ijk}\theta^j A^k_\mu \tag{1.75}$$

For the kinetic term in A^i_μ it is inappropriate to take simply the four-dimensional curl since

$$\delta\left(\partial_\mu A^i_\mu - \partial_\nu A^i_\mu\right) = c_{ijk}\theta^i\left(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu\right)$$
$$+ c_{ijk}\left[\left(\partial_\mu\theta^j\right)A^k_\nu - \left(\partial_\nu\theta^j\right)A^k_\mu\right] \tag{1.76}$$

whereas the transformation property required is

$$\delta F^i_{\mu\nu} = c_{ijk}\theta^j F^k_{\mu\nu} \tag{1.77}$$

Thus $F^i_{\mu\nu}$ must contain an additional piece and the appropriate choice turns out to be

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A^j_\mu A^k_\nu \tag{1.78}$$

To confirm this choice, one needs to evaluate

$$g c_{ijk}\delta\left(A^j_\mu A^k_\nu\right) = -c_{ijk}\left[\left(\partial_\mu\theta^j\right)A^k_\nu - \left(\partial_\nu\theta^j\right)A^k_\mu\right] + g\left(c_{ijk}c_{jlm}\theta^l A^m_\mu A^k_\nu + c_{ijk}A^j_\mu c_{klm}\theta^l A^m_\nu\right) \tag{1.79}$$

The term in parentheses on the right-hand side may be simplified by noting that an $n \times n$ matrix representation of the gauge algebra is provided, in terms of the structure constants, by

$$\left(\lambda^i\right)_{jk} = -i c_{ijk} \tag{1.80}$$

Using this, we may rewrite the last term as

$$g A^m_\mu A^n_\nu \theta^j (c_{ipn}c_{pjm} + c_{imp}c_{pjn}) = g A^m_\mu A^n_\nu \theta^j \left[\lambda^i, \lambda^j\right]_{mn} \tag{1.81}$$

$$= i g A^m_\mu A^n_\nu \theta^j c_{ijk}\lambda^k_{mn} \tag{1.82}$$

$$= g A^m_\mu A^n_\nu \theta^j c_{ijk}c_{kmn} \tag{1.83}$$

Collecting these results, we deduce that

$$\delta F^i_{\mu\nu} = \delta\left(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk}A^j_\mu A^k_\nu\right) \tag{1.84}$$

$$= c_{ijk}\theta^j\left(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu + g c_{klm}A^l_\mu A^m_\nu\right) \tag{1.85}$$

$$= c_{ijk}\theta^j F^k_{\mu\nu} \tag{1.86}$$

as required. From this it follows that

$$\delta\left(F^i_{\mu\nu}F^i_{\mu\nu}\right) = 2c_{ijk}F^i_{\mu\nu}\theta^j F^k_{\mu\nu} \tag{1.87}$$

$$= 0 \tag{1.88}$$

so we may use $-\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu}$ as the kinetic term.

To summarize these results for construction of a Yang–Mills Lagrangian: Start with a globally gauge-invariant Lagrangian

$$\mathscr{L}(\phi_k, \partial_\mu \phi_k) \tag{1.89}$$

then introduce A^i_μ ($i = 1, \ldots, n$, where the gauge group has n generators). Define

$$D_\mu \phi_k = \left(\partial_\mu - igA^i_\mu \lambda^i\right)\phi_k \tag{1.90}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + gc_{ijk}A^j_\mu A^k_\nu \tag{1.91}$$

The transformation properties are ($A_\mu = A^i_\mu \lambda^i$)

$$\phi' = \Omega\phi \tag{1.92}$$

$$A'_\mu = \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1} \tag{1.93}$$

The required Lagrangian is

$$\mathscr{L}(\phi_k, D_\mu \phi_k) - \frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} \tag{1.94}$$

When the gauge group is a direct product of two or more subgroups, a different coupling constant g may be associated with each subgroup. For example, in the simplest renormalizable model for weak interactions, the Weinberg–Salam model, the gauge group is $SU(2) \times U(1)$ and there are two independent coupling constants, as discussed later.

Before proceeding further, we give a more systematic derivation of the locally gauge invariant $\mathscr{L}$, following the analysis of Utiyama [6] (see also Glashow and Gell-Mann [7]). In what follows we shall, first, deduce the forms of $D_\mu \phi_k$ and $F^i_{\mu\nu}$ (merely written down above), and second, establish a formalism that could be extended beyond quantum electrodynamics and Yang–Mills theory to general relativity.

The questions to consider are, given a Lagrangian

$$\mathscr{L}(\phi_k, \partial_\mu \phi_k) \tag{1.95}$$

invariant globally under a group G with n independent constant parameters θ^i, then, to extend the invariance to a group G' dependent on local parameters $\theta^i(x)$:

1. What new (gauge) fields $A^p(x)$ must be introduced?
2. How does $A^p(x)$ transform under G'?
3. What is the form of the interaction?
4. What is the new Lagrangian?

We are given the global invariance under

$$\delta\phi_k = -iT^i_{kl}\theta^i\phi_l \tag{1.96}$$

with $i = 1, 2, \ldots, n$ and T^i satisfying

$$\left[T^i, T^j\right] = ic_{ijk}T^k \tag{1.97}$$

where

$$c_{ijk} = -c_{jik} \tag{1.98}$$

and

$$c_{ijl}c_{lkm} + c_{jkl}c_{lim} + c_{kil}c_{ljm} = 0 \tag{1.99}$$

Using Noether's theorem, one finds the n conserved currents

$$J^i_\mu = \frac{\partial\mathscr{L}}{\partial\phi_k}T^i_{kl}\partial_\mu\phi_l \tag{1.100}$$

$$\partial_\mu J^i_\mu = 0 \tag{1.101}$$

These conservation laws provide a necessary and sufficient condition for the invariance of $\mathscr{L}$.

Now consider

$$\delta\phi_k = -iT^i_{kl}\theta^i(x)\phi_l(x) \tag{1.102}$$

This local transformation does not leave $< J$ invariant:

$$\delta\mathscr{L} = -i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}T^i_{kl}\phi_l\partial_\mu\partial_i \tag{1.103}$$

$$\neq 0 \tag{1.104}$$

Hence it is necessary to add new fields A'^p $(p = 1, \ldots, M)$ in the Lagrangian, which we write as

$$\mathscr{L}(\phi_k, \partial_\mu\phi_k) \to \mathscr{L}'\left(\phi_k, \partial_\mu\phi_k, A'^p\right) \tag{1.105}$$

Let the transformation of A'^p be

$$\delta A'^p = U^i_{pq}\theta^i A'^q + \frac{1}{g}C^{jp}_\mu\partial_\mu\phi^j \tag{1.106}$$

Then the requirement is

$$\delta\mathscr{L} = \left[-i\frac{\partial\mathscr{L}'}{\partial\phi_k}T^j_{kl}\phi_l - i\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}T^j_{kl}\partial_\mu\phi_l + \frac{\partial\mathscr{L}'}{\partial A'^p}U^j_{pq}A'^p\right]\theta^j$$

$$+\left[-i\frac{\partial\mathscr{L}'}{\partial(\partial_\mu\phi_k)}T^j_{kl}\phi_l + \frac{1}{g}\frac{\partial\mathscr{L}'}{\partial A'^p}C^{pj}_\mu\right]\partial_\mu\theta^j \tag{1.107}$$

$$= 0 \tag{1.108}$$

Since θ^j and $\partial_\mu\theta^j$ are independent, the coefficients must vanish separately. For the coefficient of $\partial_\mu\theta^i$, this gives $4n$ equations involving A'^p and hence to determine the A' dependence uniquely, one needs $4n$ components. Further, the matrix C^{pj}_μ must be nonsingular and possess an inverse

$$C^{jp}_\mu C^{-1jq}_\mu = \delta_{pq} \tag{1.109}$$

$$C^{jp}_\mu C^{-1j'p}_\mu = g_{\mu\nu}\delta_{jj'} \tag{1.110}$$

Now we define

$$A^j_\mu = -gC^{-1jp}_\mu A'^p \tag{1.111}$$

Then

$$i\frac{\partial\mathscr{L}'}{\partial(\partial_\mu\phi_k)}T^i_{kl}\phi_l + \frac{\partial\mathscr{L}'}{\partial A^i_\mu} = 0 \tag{1.112}$$

so only the combination

$$D_\mu\phi_k = \partial_\mu\phi_k - iT^i_{kl}\phi_l A^i_\mu \tag{1.113}$$

occurs in the Lagrangian

$$\mathscr{L}'\left(\phi_k, \partial_\mu\phi_k, A'^p\right) = \mathscr{L}''(\phi_k, D_\mu\phi_k) \tag{1.114}$$

It follows from this equality of $\mathscr{L}'$ and $\mathscr{L}''$ that

$$\left.\frac{\partial\mathscr{L}''}{\partial\phi_k}\right|_{D_\mu\phi} - i\left.\frac{\partial\mathscr{L}''}{\partial(D_\mu\phi_l)}\right|_\phi T^i_{kl}A^i_\mu = \frac{\partial\mathscr{L}'}{\partial\phi_k} \tag{1.115}$$

$$\left.\frac{\partial\mathscr{L}''}{\partial(D_\mu\phi_k)}\right|_\phi = \frac{\partial\mathscr{L}'}{\partial(D_\mu\phi_k)} \tag{1.116}$$

$$ig\left.\frac{\partial\mathscr{L}''}{\partial(D_\mu\phi_k)}\right|_\phi T^a_{kl}\phi_l C^{-1ap}_\mu = \frac{\partial\mathscr{L}'}{\partial A'^p} \tag{1.117}$$

These relations may be substituted into the vanishing coefficient of θ^j occurring in $\delta\mathscr{L}'$ (above). The result is

$$0 = -i\left[\frac{\partial \mathscr{L}''}{\partial \phi_k}\bigg|_{D_\mu \phi} T^i_{kl}\phi_l + \frac{\partial \mathscr{L}''}{\partial (D_\mu \phi_k)}\bigg|_{\phi} T^i_{kl} D_\mu \phi_l\right]$$

$$+ i \frac{\partial \mathscr{L}''}{\partial \phi_k}\bigg|_{\phi} \left(\phi_l A^a_\nu \left\{ i \left[T^a, T^i\right]_{kl} \lambda_{\mu\nu} + S^{ba,j}_{\mu\nu}\right\}\right) = 0 \tag{1.118}$$

where

$$S^{ba,j}_{\mu\nu} = C^{-1ap}_{\mu} U^j_{pq} C^{bq}_{\nu} \tag{1.119}$$

is defined such that

$$\delta A^a_\mu = g\delta\left(-C^{-1ap}_{\mu} A'^p\right) \tag{1.120}$$

$$= S^{ba,j}_{\mu\nu} A^b_\nu \theta^j - \frac{1}{g}\partial_\mu \theta^a \tag{1.121}$$

Now the term in the first set of brackets in Eq. (1.118) vanishes if we make the identification

$$\mathscr{L}''(\phi_k, D_\mu \phi_k) = \mathscr{L}(\phi_k, D_\mu \phi_k) \tag{1.122}$$

The vanishing of the final term in parentheses in Eq. (1.118) then enables us to identify

$$S^{ba,j}_{\mu\nu} = -c_{ajb} g_{\mu\nu} \tag{1.123}$$

It follows that

$$\delta A^a_\mu = c_{abc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu \theta^a \tag{1.124}$$

From the transformations δA^A_μ and $\delta \phi_k$, one can show that

$$\delta(D_\mu \phi_k) = \delta\left(\partial_\mu \phi_k - i T^a_{kl} A^a_\mu \phi_l\right) \tag{1.125}$$

$$= -i T^i_{kl} \theta^i (D_\mu \phi_l) \tag{1.126}$$

This shows that $D_\mu \phi_k$ transforms covariantly.

Let the Lagrangian density for the free A^a_μ field be

$$\mathscr{L}_0\left(A^a_\mu, \partial_\nu A^a_\mu\right) \tag{1.127}$$

Using

$$\delta A^a_\mu = c_{abc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu \theta^a \tag{1.128}$$

one finds (from $\delta\mathscr{L} = 0$)

$$\frac{\partial \mathscr{L}_0}{\partial A^a_\mu} c_{abc} A^c_\mu + \frac{\partial \mathscr{L}_0}{\partial(\partial_\nu A^a_\mu)} c_{abc} \partial_\nu A^c_\mu = 0 \tag{1.129}$$

$$-\frac{\partial \mathscr{L}_0}{\partial A^a_\mu} + \frac{\partial \mathscr{L}_0}{\partial(\partial_\mu A^b_\nu)} c_{abc} A^c_\nu = 0 \tag{1.130}$$

$$\frac{\partial \mathscr{L}_0}{\partial(\partial_\nu A^a_\mu)} + \frac{\partial \mathscr{L}_0}{\partial(\partial_\mu A^a_\mu)} + \frac{\partial \mathscr{L}_0}{\partial(\partial_\mu A^a_\nu)} = 0 \tag{1.131}$$

From the last of these three it follows that $\partial_\mu A^a_\mu$ occurs only in the antisymmetric combination

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \tag{1.132}$$

Using the preceding equation then gives

$$\frac{\partial \mathscr{L}_0}{\partial A^a_\mu} = \frac{\partial \mathscr{L}_0}{\partial A^b_{\mu\nu}} c_{abc} A^c_\nu \tag{1.133}$$

so the only combination occurring is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g c_{abc} A^b_\mu A^c_\nu \tag{1.134}$$

Thus, we may put

$$\mathscr{L}_0\big(A^a_\mu, \partial_\nu A^a_\mu\big) = \mathscr{L}'_0\big(A^a_\mu, F^a_{\mu\nu}\big) \tag{1.135}$$

Then

$$\left.\frac{\partial \mathscr{L}_0}{\partial(\partial_\nu A^a_\mu)}\right|_A = \left.\frac{\partial \mathscr{L}'_0}{\partial F^a_{\mu\nu}}\right|_A \tag{1.136}$$

$$\left.\frac{\partial \mathscr{L}_0}{\partial A^a_\mu}\right|_{\partial_\mu A} = \left.\frac{\partial \mathscr{L}'_0}{\partial A^a_\mu}\right|_F + \left.\frac{\partial \mathscr{L}'_0}{\partial(\partial F^b_{\mu\nu})}\right|_A c_{abc} A^c_\nu \tag{1.137}$$

But one already knows that

$$\left.\frac{\partial \mathscr{L}_0}{\partial A^a_\mu}\right|_{\partial_\mu A} = \frac{\partial \mathscr{L}'_0}{\partial F^b_{\mu\nu}} c_{abc} A^c_\nu \tag{1.138}$$

and it follows that $\mathscr{L}'_0$ does not depend explicitly on A^a_μ.

$$\mathscr{L}_0(A_\mu, \partial_\nu A_\mu) = \mathscr{L}''_0\big(F^a_{\mu\nu}\big) \tag{1.139}$$

Bearing in mind both the analogy with quantum electrodynamics and renormalizability we write

$$\mathscr{L}_0''\left(F^a_{\mu\nu}\right) = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \tag{1.140}$$

When all structure constants vanish, this then reduces to the usual Abelian case. The final Lagrangian is therefore

$$\mathscr{L}(\phi_k, D_\mu\phi_k) - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \tag{1.141}$$

Defining matrices M^i in the adjoint representation by

$$M^i_{ab} = -ic_{iab} \tag{1.142}$$

the transformation properties are

$$\delta\phi_k = -iT^i_{kl}\theta^i\phi_l \tag{1.143}$$

$$\delta A^a_\mu = -iM^i_{ab}\theta^i A^b_\mu - \frac{1}{g}\partial_\mu\theta^a \tag{1.144}$$

$$\delta(D_\mu\phi_k) = -iT_{kl}\theta^i(D_\mu\phi_l) \tag{1.145}$$

$$\delta F^a_{\mu\nu} = -M^i_{ab}\theta^i F^b_{\mu\nu} \tag{1.146}$$

Clearly, the Yang–Mills theory is most elegant when the matter fields are in the adjoint representation like the gauge fields because then the transformation properties of ϕ_k, $D_\mu\phi_k$ and $F^a_{\mu\nu}$ all coincide. But in theories of physical interest for strong and weak interactions, the matter fields will often, instead, be put into the fundamental representation of the gauge group.

Let us give briefly three examples, the first Abelian and the next two non-Abelian.

Example 1 (Quantum Electrodynamics). For free fermions

$$\mathscr{L}\bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1.147}$$

the covariant derivative is

$$D_\mu\psi = \partial_\mu\psi + ieA_\mu\psi \tag{1.148}$$

This leads to

$$\mathscr{L}(\psi, D_\mu\psi) - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{1.149}$$

Example 2 (Scalar ϕ^4 Theory with ϕ^a in Adjoint Representation). The globally invariant Lagrangian is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi^a\partial - \mu\phi^a - \frac{1}{2}\mu^2\phi^a\phi^a - \frac{1}{4}\lambda\left(\phi^a\phi^a\right)^2 \tag{1.150}$$

One introduces

$$D_\mu\phi^a = \partial_\mu\phi^a - gc_{abc}A^b\mu\phi^c \tag{1.151}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc_{ac}A^b_\mu A^c_\nu \tag{1.152}$$

and the appropriate Yang–Mills Lagrangian is then

$$\mathscr{L} = \frac{1}{2}\left(D_\mu\phi^a\right)\left(D_\mu\phi^a\right) - \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2}\mu^a\phi^a\phi^a - \frac{1}{4}\left(\phi^a\phi^a\right)^2 \tag{1.153}$$

Example 3 (Quantum Chromodynamics). Here the quarks ψ_k are in the fundamental (three-dimensional) representation of SU(3). The Lagrangian for free quarks is

$$\mathscr{L}\bar{\psi}_k(i\partial\!\!\!/ - m)\psi_k \tag{1.154}$$

We now introduce

$$D_\mu\psi_k = \partial_\mu\psi_k - \frac{1}{2}g\lambda^i_{kl}A^i_\mu\psi_l \tag{1.155}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc}A^b_\mu A^c_\nu \tag{1.156}$$

and the appropriate Yang–Mills Lagrangian is

$$\mathscr{L}\bar{\psi}(iD\!\!\!\!/ - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^a\mu\nu \tag{1.157}$$

If a flavor group (which is *not* gauged) is introduced, the quarks carry an additional label ψ^a_k, and the mass term becomes a diagonal matrix $m \to -M_a\delta_{ab}$.

The advantage of this Utiyama procedure is that it may be generalized to include general relativity (see Utiyama [6], Kibble [8], and more recent works [9–12]).

Finally, note that any mass term of the form $+m_i^2A^i_\mu A^i_\mu$ will violate the local gauge invariance of the Lagrangian density $\mathscr{L}$. From what we have stated so far, the theory must contain n massless vector particles, where n is the number of generators of the gauge group; at least, this is true as long as the local gauge symmetry is unbroken.

1.4 Nambu–Goldstone Conjecture

We have seen that the imposition of a non-Abelian local gauge invariance appears to require the existence of a number of massless gauge vector bosons equal to the number of generators of the gauge group; this follows from the fact that a mass term $+\frac{1}{2}^2A^i_\mu A^i_\mu$ in $\mathscr{L}$ breaks the local invariance. Since in nature only one massless spin-1 particle—the photon—is known, it follows that if we are to exploit a local gauge group less trivial than U(1), the symmetry must be broken.

Let us therefore recall two distinct ways in which a symmetry may be broken. If there is exact symmetry, this means that under the transformations of the group the Lagrangian is invariant:

$$\delta \mathscr{L} = 0 \tag{1.158}$$

Further, the vacuum is left invariant under the action of the group generators (charges) Q_i:

$$Q_i |0\rangle = 0 \tag{1.159}$$

From this, it follows that all the Q_i commute with the Hamiltonian

$$\left[Q^i, H\right] = 0 \tag{1.160}$$

and that the particle multiplets must be mass degenerate.

The first mechanism to be considered is explicit symmetry breaking, where one adds to the symmetric Lagrangian ($\mathscr{L}_0$) a piece ($\mathscr{L}_1$) that is noninvariant under the full symmetry group G, although $\mathscr{L}_1$ may be invariant under some subgroup G' of G. Then

$$\mathscr{L} = \mathscr{L}_0 = c\mathscr{L}_1 \tag{1.161}$$

and under the group transformation,

$$\delta \mathscr{L}_0 = 0 \tag{1.162}$$

$$\delta \mathscr{L}_1 \neq 0 \tag{1.163}$$

while

$$Q_i |0\rangle \to 0 \quad \text{as } c \to 0 \tag{1.164}$$

The explicit breaking is used traditionally for the breaking of flavor groups SU(3) and SU(4) in hadron physics.

The second mechanism is spontaneous symmetry breaking (perhaps more appropriately called *hidden symmetry*). In this case the Lagrangian is symmetric,

$$\delta \mathscr{L} = 0 \tag{1.165}$$

but the vacuum is not:

$$Q_i |0\rangle \neq 0 \tag{1.166}$$

This is because as a consequence of the dynamics the vacuum state is degenerate, and the choice of one as the physical vacuum breaks the symmetry. This leads to nondegenerate particle multiplets.

It is possible that both explicit and spontaneous symmetry breaking be present. One then has

$$\mathscr{L} = \mathscr{L}_0 + c\mathscr{L}_1 \tag{1.167}$$

$$\delta L = 0 \tag{1.168}$$

$$\delta L_1 \neq 0 \tag{1.169}$$

but

$$Q^i|0\rangle \neq 0 \quad \text{as } c \to 0 \tag{1.170}$$

An example that illustrates all of these possibilities is the infinite ferromagnet, where the symmetry in question is rotational invariance. In the paramagnetic phase at temperature $T > T_c$ there is exact symmetry; in the ferromagnetic phase, $T < T_c$, there is spontaneous symmetry breaking. When an external magnetic field is applied, this gives explicit symmetry breaking for both $T > T_c$ and $T < T_c$.

Here we are concerned with Nambu and Goldstone's well-known conjecture [13–15] that when there is spontaneous breaking of a continuous symmetry in a quantum field theory, there must exist massless spin-0 particles. If this conjecture were always correct, the situation would be hopeless. Fortunately, although the Nambu–Goldstone conjecture applies to global symmetries as considered here, the conjecture fails for local gauge theories because of the Higgs mechanism described in Section 1.5.

It is worth remarking that in the presence of spontaneous breakdown of symmetry the usual argument of Noether's theorem that leads to a conserved charge breaks down. Suppose that the global symmetry is

$$\phi_k \to \phi_k - iT^i_{kl}\phi_l\theta^i \tag{1.171}$$

Then

$$\partial_\mu j^i_\mu = 0 \tag{1.172}$$

$$J^i_\mu = -i\left[\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_k)}T^i_{kl}\phi_l\right] \tag{1.173}$$

but the corresponding *charge*,

$$Q^i = \int d^3x j^i_0 \tag{1.174}$$

will not exist because the current does not fall off sufficiently fast with distance to make the integral convergent.

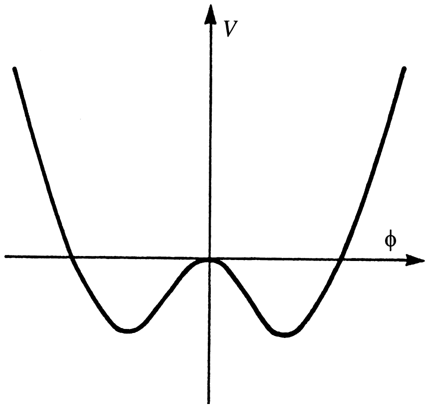

Figure 1.1 Potential function $V(\phi)$.

The simplest model field theory [14] to exhibit spontaneous symmetry breaking is the one with Lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi\partial_\mu\phi - m_0^2\phi^2\right) - \frac{\lambda_0}{24}\phi^4 \tag{1.175}$$

For $m_0^2 > 0$, one can apply the usual quantization procedures, but for $m_0^2 < 0$, the potential function

$$V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{24}\phi^4 \tag{1.176}$$

has the shape depicted in Fig. 1.1. The ground state occurs where $V'(\phi_a) = 0$, corresponding to

$$\phi_0 = \pm\chi = \pm\sqrt{\frac{-6m_0^2}{\lambda_0}} \tag{1.177}$$

Taking the positive root, it is necessary to define a shifted field ϕ' by

$$\phi = \phi' + \chi \tag{1.178}$$

Inserting this into the Lagrangian $\mathscr{L}$ leads to

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi'\partial_\mu\phi' + 2m_0^2\phi'^2\right) - \frac{1}{6}\lambda_0\chi\phi'^3 - \frac{\lambda_0}{24}\phi'^4 + \frac{3m_0^4}{\lambda_0} \tag{1.179}$$

The (mass)^2 of the ϕ' field is seen to be $-2m_0^2 < 0$, and this Lagrangian may now be treated by canonical methods. The symmetry $\phi \to -\phi$ of the original Lagrangian has disappeared. We may choose either of the vacuum states $\phi = \pm\chi$ as the physical vacuum without affecting the theory, but once a choice of vacuum is made, the reflection symmetry is lost. Note that the Fock spaces built on the two possi-