13 – Gauge Invariance

Topics: Time-dependent potentials and fields, gauge transformations

Summary: Students are first reminded of why EM fields can be written in terms of a scalar and vector potential. They then show that a gauge transformation in the vector potential results in an identical magnetic field. Students derive an integral relationship between **E** & **A**, and then find the necessary conditions for transforming V. A challenge question asks students to derive Poisson's equation, which would be used to solve for the scalar function λ that transforms the potentials to the Coulomb gauge.

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Comments: Students should be able to complete these tasks in less than 50 minutes. The biggest complaints from students has been about not entirely understanding why we would want to transform the potentials in the first place. This is hinted at in the final challenge question, where an equation is found for the function that transforms to the Coulomb gauge, but is not explicitly addressed here. The first task is a review intended to orient students to the remaining tasks, reminding them of why we can write the fields in terms of potentials. Many students have forgotten that the various statements that can be made about divergenceless (or irrotational) fields are all equivalent. [See Sect. 1.6.2 in Griffiths] Some students were unsure about the cross product being a linear operator (whether the curl of the sum of two vectors is equal to the sum of the curls). There has been a modest amount of confusion in keeping track of primed and unprimed quantities, with students sometimes mixing them up in their heads. The challenge question is interesting because we rarely have students think about the actual procedure for finding the function that transforms the potentials to a specific gauge – it's usually sufficient to know simply that the transformation can always be done in principle. The challenge question may be helpful in making the origins of this scalar field less mysterious.

A. The following identities (I & II) are true for **any** vector field $\vec{\mathbf{F}}(\vec{\mathbf{r}},t)$ or scalar function $f(\vec{\mathbf{r}},t)$:

I. $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{\mathbf{F}}\right)=0$

II. $\vec{\nabla}\times\left(\vec{\nabla}f\right)=0$

Maxwell's *time-independent* equations are:

$\vec{\nabla}\cdot\vec{\mathbf{E}}=\rho/\varepsilon_0$ $\vec{\nabla}\times\vec{\mathbf{E}}=0$ $\vec{\nabla}\cdot\vec{\mathbf{B}}=0$ $\vec{\nabla}\times\vec{\mathbf{B}}=\mu_0\vec{\mathbf{J}}$

Given the information above about scalar and vector fields, which of these Maxwell equations implies that $\vec{\mathbf{E}}=-\vec{\nabla}V$? Explain your reasoning.

Which of these Maxwell equations implies that $\vec{\mathbf{B}}=\vec{\nabla}\times\vec{\mathbf{A}}$? Explain your reasoning.

B. Recall the second math identity from the first page:

II. $\vec{\nabla}\times\left(\vec{\nabla}f\right)=0$

$\vec{\nabla}\cdot\vec{\mathbf{B}}=0$ says the magnetic field can be written as the curl of some vector potential $\vec{\mathbf{A}}$:

$$\vec{\mathbf{B}}=\vec{\nabla}\times\vec{\mathbf{A}}$$

Use Eq. II above to show that if we create a *new* vector potential $\vec{\mathbf{A}}'$, gotten by adding $\vec{\nabla}\lambda$ (where $\lambda\left(\vec{r},t\right)$ is some scalar function of space and time) to the first vector potential $\vec{\mathbf{A}}$:

$$\vec{\mathbf{A}}'=\vec{\mathbf{A}}+\vec{\nabla}\lambda$$

then this new vector potential will also correspond to the same B-field:

$$\vec{\mathbf{B}}=\vec{\nabla}\times\vec{\mathbf{A}}'$$

C. Faraday's Law in differential form is

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Stokes' theorem says that the following integral relationship is also true:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\ell} = -\frac{d}{dt} \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

What would be an analogous integral expression involving the vector potential $\vec{\mathbf{A}}$ and the magnetic field $\vec{\mathbf{B}}$, given that $\vec{\nabla} \times \vec{\mathbf{A}} = \vec{\mathbf{B}}$?

Combine this integral relationship between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ from above with Faraday's law in integral form to find an integral relationship between $\vec{\mathbf{E}}$ and $\vec{\mathbf{A}}$.

D. If $\oint \vec{\mathbf{E}} \cdot d\vec{\ell} = 0$ says that $\vec{\mathbf{E}} = -\vec{\nabla} V$, use the integral relationship you derived on the previous page between $\vec{\mathbf{E}}$ & $\vec{\mathbf{A}}$ to show that

$$\vec{\mathbf{E}}' + \frac{\partial \vec{\mathbf{A}}'}{\partial t} = -\vec{\nabla} V'$$

Is this relationship also true in time-*independent* situations? Why or why not?

Suppose we have two vector potentials $\vec{\mathbf{A}}$ & $\vec{\mathbf{A}}'$, such that

$$\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\nabla}\lambda$$

Is it also true that $\oint \vec{\mathbf{A}} \cdot d\vec{\ell} = \oint \vec{\mathbf{A}}' \cdot d\vec{\ell}$? Why or why not?

E. Suppose the following two relationships are true:

$$\vec{\mathbf{E}} = -\vec{\nabla} V - \frac{\partial \vec{\mathbf{A}}}{\partial t} \qquad \& \qquad \vec{\mathbf{E}}' = -\vec{\nabla} V' - \frac{\partial \vec{\mathbf{A}}'}{\partial t}$$

where $\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\nabla}\lambda$.

If we know that both expressions correspond to the same electric field ($\vec{\mathbf{E}} = \vec{\mathbf{E}}'$), what is the relationship between V and V'?

Challenge Question: Imagine we know the divergence of $\vec{\mathbf{A}}$ is equal to some scalar function $\vec{\nabla} \cdot \vec{\mathbf{A}} = f(\vec{\mathbf{r}}, t)$ and we want to construct a new vector potential so that $\vec{\nabla} \cdot \vec{\mathbf{A}}' = 0$.

Use the fact that $\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\nabla}\lambda$ to find a differential equation for the scalar function $\lambda(\vec{r}, t)$, so that the electric and magnetic fields are unchanged and $\vec{\nabla} \cdot \vec{\mathbf{A}}' = 0$ everywhere and at all times.