Riemannian Curvature

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We now generalize our computation of curvature to arbitrary spaces.

1 Parallel transport around a small closed loop

We compute the change in a vector, w^α, which we parallel transport around a closed loop. The loop starts at a point $\mathcal{P}$, which we take to have coordinates $(0,0)$, progresses along a direction u^α a distance λ to a point at $(\lambda,0)$. Next, we transport along the direction v^α a distance σ to a point at (λ,σ), giving a curve C_1 from $(0,0)$ to (λ,σ). Rather than returning in the direction $-u^\alpha$ by λ then $-v^\alpha$ by σ, we perform a second transport, interchanging the order: first v^α by σ, then u^α by λ, giving curve C_2 from $(0,0)$ to (λ,σ). We then compare the expressions for $w^\alpha(\lambda,\sigma)$ along the two curves. This gives the same result as if we had transported around the closed loop $C_1 - C_2$, but the computation is easier this way.

Start with two directions, $u^\alpha(0,0)$ and $v^\alpha(0,0)$. Parallel transport each in the direction of the other to get $u^\alpha(0,\sigma)$ and $v^\alpha(\lambda,0)$:

$$
\begin{aligned}
v^\alpha(\lambda,0) &= v^\alpha(0,0)+\frac{dv^\alpha}{d\lambda}(0,0)\,\lambda \\
&= v^\alpha(0,0)-u^\beta(0,0)\,v^\mu(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
u^\alpha(0,\sigma) &= u^\alpha(0,0)+\frac{du^\alpha}{d\lambda}(0,0)\,\sigma \\
&= u^\alpha(0,0)-v^\beta(0,0)\,u^\mu(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\sigma
\end{aligned}
$$

Now take a third vector, $w^\alpha(0,0)$ and transport it to $w^\alpha(\lambda,\sigma)$ in two different ways, first along u^α and then v^α, then in the opposite order.

First,

$$
w^\alpha(\lambda,0)=w^\alpha(0,0)-u^\beta(0,0)\,w^\mu(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\lambda
$$

then

$$
\begin{aligned}
w^\alpha(\lambda,\sigma) &= w^\alpha(\lambda,0)-v^\beta(\lambda,0)\,w^\mu(\lambda,0)\,\Gamma^\alpha_{\mu\beta}(\lambda,0)\,\sigma \\
&= \left(w^\alpha(0,0)-u^\beta(0,0)\,w^\mu(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\lambda\right) \\
&\quad -\left(v^\beta(0,0)-u^\lambda(0,0)\,v^\tau(0,0)\,\Gamma^\beta_{\tau\lambda}(0,0)\,\lambda\right)\left(w^\mu(0,0)-u^\rho(0,0)\,w^\sigma(0,0)\,\Gamma^\mu_{\sigma\rho}(0,0)\,\lambda\right)\left(\Gamma^\alpha_{\mu\beta}(0,0)+u^\delta\partial_\delta\Gamma^\alpha_{\mu\beta}\right. \\
&= w^\alpha(0,0) \\
&\quad -u^\beta(0,0)\,w^\mu(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\lambda \\
&\quad -v^\beta(0,0)\,w^\mu(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\sigma \\
&\quad +u^\lambda(0,0)\,v^\tau(0,0)\,w^\mu(0,0)\,\Gamma^\beta_{\tau\lambda}(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\sigma\lambda \\
&\quad +v^\beta(0,0)\,u^\rho(0,0)\,w^\sigma(0,0)\,\Gamma^\mu_{\sigma\rho}(0,0)\,\Gamma^\alpha_{\mu\beta}(0,0)\,\sigma\lambda
\end{aligned}
$$

$$-v^{\beta}(0,0)\,w^{\mu}(0,0)\,u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}(0,0)\,\lambda\sigma$$
$$+u^{\lambda}(0,0)\,v^{\tau}(0,0)\,w^{\mu}(0,0)\,\Gamma^{\beta}_{\tau\lambda}(0,0)\,u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}(0,0)\,\lambda^{2}\sigma$$
$$+v^{\beta}(0,0)\,u^{\rho}(0,0)\,w^{\sigma}(0,0)\,\Gamma^{\mu}_{\sigma\rho}(0,0)\,u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}(0,0)\,\lambda^{2}\sigma$$
$$-u^{\lambda}(0,0)\,v^{\tau}(0,0)\,\Gamma^{\beta}_{\tau\lambda}(0,0)\,u^{\rho}(0,0)\,w^{\sigma}(0,0)\,\Gamma^{\mu}_{\sigma\rho}(0,0)\,\Gamma^{\alpha}_{\mu\beta}(0,0)\,\lambda^{2}\sigma$$
$$-\left(u^{\lambda}(0,0)\,v^{\tau}(0,0)\,\Gamma^{\beta}_{\tau\lambda}(0,0)\,u^{\rho}(0,0)\,w^{\sigma}(0,0)\,\Gamma^{\mu}_{\sigma\rho}(0,0)\right)u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}(0,0)\,\lambda^{3}\sigma$$

Now that all quantities are expressed at $\mathcal{P}=(0,0)$, we may drop these $(0,0)$ arguments. Keeping only terms to second order:

$$\begin{aligned} w^{\alpha}_{C_1}(\lambda,\sigma) &= w^{\alpha}-u^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\lambda-v^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\sigma \\ &\quad -v^{\beta}w^{\mu}u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}\lambda\sigma+u^{\lambda}v^{\tau}w^{\mu}\Gamma^{\beta}_{\tau\lambda}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda \\ &\quad +v^{\beta}u^{\rho}w^{\sigma}\Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda \end{aligned}$$

Now repeat in the opposite order (just interchange u^{α} with v^{α} and σ with λ),

$$\begin{aligned} w^{\alpha}_{C_2}(\lambda,\sigma) &= w^{\alpha}-v^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\sigma-u^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\lambda \\ &\quad -u^{\beta}w^{\mu}v^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}\lambda\sigma+v^{\lambda}u^{\tau}w^{\mu}\Gamma^{\beta}_{\tau\lambda}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda \\ &\quad +u^{\beta}v^{\rho}w^{\sigma}\Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda \end{aligned}$$

The difference between these gives the change in w^{α} under parallel transport around the loop

$$\begin{aligned} \delta w^{\alpha} &= w^{\alpha}_{C_1-C_2} \\ &= w^{\alpha}_{C_1}(\lambda,\sigma)-w^{\alpha}_{C_2}(\lambda,\sigma) \\ &= w^{\alpha}-u^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\lambda-v^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\sigma+u^{\lambda}v^{\tau}w^{\mu}\Gamma^{\beta}_{\tau\lambda}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda \\ &\quad +v^{\beta}u^{\rho}w^{\sigma}\Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda-v^{\beta}w^{\mu}u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}\lambda\sigma \\ &\quad -w^{\alpha}+v^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\sigma+u^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\lambda-v^{\lambda}u^{\tau}w^{\mu}\Gamma^{\beta}_{\tau\lambda}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda \\ &\quad -u^{\beta}v^{\rho}w^{\sigma}\Gamma^{\mu}_{\sigma\rho}\Gamma^{\alpha}_{\mu\beta}\sigma\lambda+u^{\beta}w^{\mu}v^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}\lambda\sigma \end{aligned}$$

so

$$\begin{aligned} \delta w^{\alpha} &= w^{\alpha}-w^{\alpha}-u^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\lambda+u^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\lambda-v^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\sigma+v^{\beta}w^{\mu}\Gamma^{\alpha}_{\mu\beta}\sigma \\ &\quad +w^{\mu}\left(u^{\beta}v^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}-v^{\beta}u^{\delta}\partial_{\delta}\Gamma^{\alpha}_{\mu\beta}\right)\sigma\lambda \\ &\quad +w^{\mu}\left(u^{\lambda}v^{\tau}\Gamma^{\beta}_{\tau\lambda}\Gamma^{\alpha}_{\mu\beta}+v^{\beta}u^{\rho}\Gamma^{\sigma}_{\mu\rho}\Gamma^{\alpha}_{\sigma\beta}-v^{\lambda}u^{\tau}\Gamma^{\beta}_{\tau\lambda}\Gamma^{\alpha}_{\mu\beta}-u^{\beta}v^{\rho}\Gamma^{\sigma}_{\mu\rho}\Gamma^{\alpha}_{\sigma\beta}\right)\lambda\sigma \\ &= w^{\mu}\left(u^{\beta}v^{\nu}\Gamma^{\alpha}_{\mu\beta,\nu}-v^{\nu}u^{\beta}\Gamma^{\alpha}_{\mu\nu,\beta}\right)\sigma\lambda \\ &\quad +w^{\mu}\left(u^{\lambda}v^{\tau}\left(\Gamma^{\beta}_{\tau\lambda}-\Gamma^{\beta}_{\lambda\tau}\right)\Gamma^{\alpha}_{\mu\beta}+v^{\nu}u^{\beta}\Gamma^{\sigma}_{\mu\beta}\Gamma^{\alpha}_{\sigma\nu}-u^{\beta}v^{\nu}\Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}\right)\lambda\sigma \\ &\quad +w^{\mu}\left(\Gamma^{\alpha}_{\mu\beta,\nu}-\Gamma^{\alpha}_{\mu\nu,\beta}+\Gamma^{\sigma}_{\mu\beta}\Gamma^{\alpha}_{\sigma\nu}-\Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}\right)v^{\nu}u^{\beta}\lambda\sigma \end{aligned}$$

The change in w^{α} per unit area is our measure of the curvature,

$$\begin{aligned} \frac{\delta w^{\alpha}}{\lambda\sigma} &= w^{\mu}\left(\Gamma^{\alpha}_{\mu\beta,\nu}-\Gamma^{\alpha}_{\mu\nu,\beta}+\Gamma^{\sigma}_{\mu\beta}\Gamma^{\alpha}_{\sigma\nu}-\Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}\right)v^{\nu}u^{\beta} \\ &= w^{\mu}R^{\alpha}_{\mu\beta\nu}u^{\beta}v^{\nu} \end{aligned}$$

We see that the change in w^{α} depends linearly on w^{α}, and also on the orientation of the infinitesimal loop spanned by u^{α} and v^{α}. The rest of the expression,

$$R^{\alpha}_{\mu\beta\nu}\equiv\Gamma^{\alpha}_{\mu\beta,\nu}-\Gamma^{\alpha}_{\mu\nu,\beta}+\Gamma^{\sigma}_{\mu\beta}\Gamma^{\alpha}_{\sigma\nu}-\Gamma^{\sigma}_{\mu\nu}\Gamma^{\alpha}_{\sigma\beta}$$

is called the Riemann curvature tensor. It may be thought of as a trilinear operator which takes an oriented unit area element,

$$S^{\beta\nu} = \frac{1}{2}\left(u^\beta v^\nu - u^\nu v^\beta\right)$$

to give a linear operator, $T^\alpha_{\ \mu} = R^\alpha_{\ \mu\beta\nu} S^{\beta\nu}$. This linear operator gives the infinitesimal change in w^α per unit area, in the limit of zero area, when w^α is transported around the area element with orientation $S^{\beta\nu}$.

Since all elements of this construction are defined geometrically from within the manifold, $R^\alpha_{\ \mu\beta\nu}$ is intrinsic to the manifold. To see that it is also a tensor, we could recompute the same construction in different coordinates. Since the entire construction is perturbative, we would find the components of $R^\alpha_{\ \mu\beta\nu}$ changing linearly and homogeneously in the transformation matrix. However, there is an easier way to prove that $R^\alpha_{\ \mu\beta\nu}$ is a tensor. Consider two covariant derivatives of the vector w^α,

$$\begin{aligned} D_\mu D_\nu w^\alpha &= D_\mu \left(\partial_\nu w^\alpha + w^\beta \Gamma^\alpha_{\ \beta\nu}\right) \\ &= \partial_\mu \left(\partial_\nu w^\alpha + w^\beta \Gamma^\alpha_{\ \beta\nu}\right) + \left(\partial_\nu w^\rho + w^\beta \Gamma^\rho_{\ \beta\nu}\right)\Gamma^\alpha_{\ \rho\mu} - \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right)\Gamma^\rho_{\ \nu\mu} \end{aligned}$$

Because we use covariant derivatives, this object is necessarily a tensor. Now take the derivatives in the opposite order and subtract, giving the commutator. This is also a tensor,

$$\begin{aligned} \left[D_\mu, D_\nu\right] w^\alpha &= D_\mu D_\nu w^\alpha - D_\nu D_\mu w^\alpha \\ &= \partial_\mu \partial_\nu w^\alpha + \partial_\mu w^\beta \Gamma^\alpha_{\ \beta\nu} + w^\beta \Gamma^\alpha_{\ \beta\nu,\mu} + \left(\partial_\nu w^\rho + w^\beta \Gamma^\rho_{\ \beta\nu}\right)\Gamma^\alpha_{\ \rho\mu} - \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right)\Gamma^\rho_{\ \nu\mu} \\ &\quad - \partial_\nu \partial_\mu w^\alpha - \partial_\nu w^\beta \Gamma^\alpha_{\ \beta\mu} - w^\beta \Gamma^\alpha_{\ \beta\mu,\nu} - \left(\partial_\mu w^\rho + w^\beta \Gamma^\rho_{\ \beta\mu}\right)\Gamma^\alpha_{\ \rho\nu} + \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right)\Gamma^\rho_{\ \mu\nu} \\ &= \partial_\mu \partial_\nu w^\alpha - \partial_\nu \partial_\mu w^\alpha + \partial_\mu w^\beta \Gamma^\alpha_{\ \beta\nu} + \partial_\nu w^\rho \Gamma^\alpha_{\ \rho\mu} - \partial_\nu w^\beta \Gamma^\alpha_{\ \beta\mu} - \partial_\mu w^\rho \Gamma^\alpha_{\ \rho\nu} \\ &\quad - \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right)\Gamma^\rho_{\ \nu\mu} + \left(\partial_\rho w^\alpha + w^\beta \Gamma^\alpha_{\ \rho\nu}\right)\Gamma^\rho_{\ \mu\nu} \\ &\quad + w^\beta \Gamma^\alpha_{\ \beta\nu,\mu} + w^\beta \Gamma^\rho_{\ \beta\nu}\Gamma^\alpha_{\ \rho\mu} - w^\beta \Gamma^\alpha_{\ \beta\mu,\nu} - w^\beta \Gamma^\rho_{\ \beta\mu}\Gamma^\alpha_{\ \rho\nu} \\ &= w^\beta R^\alpha_{\ \beta\nu\mu} \end{aligned}$$

Since the result is w^β properly contracted with $R^\alpha_{\ \beta\nu\mu}$, and w^β is a tensor, $R^\alpha_{\ \beta\nu\mu}$ must also be a tensor.

2 Symmetries of the curvature tensor

Recall that parallel transport of w^α preserves the length, $w^\alpha w_\alpha$ of w^α. This means that the transformation,

$$\begin{aligned} \left(\delta^\alpha_\mu + T^\alpha_{\ \mu}\sigma\lambda\right) w^\mu &= w^\alpha + w^\mu R^\alpha_{\ \mu\beta\nu} S^{\beta\nu}\sigma\lambda \\ &= w^\alpha + \delta w^\alpha \end{aligned}$$

must be an infinitesimal Lorentz transformation, $\Lambda^\alpha_{\ \beta} = \delta^\alpha_\beta + \varepsilon^\alpha_{\ \beta}$. An infinitesimal Lorentz transformation satisfies

$$\begin{aligned} \eta_{\alpha\beta} &= \eta_{\mu\nu}\Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta} \\ &= \eta_{\mu\nu}\left(\delta^\mu_\alpha + \varepsilon^\mu_{\ \alpha}\right)\left(\delta^\nu_\beta + \varepsilon^\nu_{\ \beta}\right) \\ &= \eta_{\alpha\beta} + \eta_{\alpha\nu}\varepsilon^\nu_{\ \beta} + \eta_{\mu\beta}\varepsilon^\mu_{\ \alpha} + \mathcal{O}\left(\varepsilon^2\right) \\ 0 &= \varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha} \end{aligned}$$

so when the indices of an infinitesimal Lorentz transformation are both in the same position, they must be antisymmetric. Therefore, $T_{\alpha\beta} = -T_{\beta\alpha}$ and we must have

$$R_{\alpha\beta\mu\nu} S^{\mu\nu} = -R_{\beta\alpha\mu\nu} S^{\mu\nu}$$

for any surface element $S^{\mu\nu}$. Therefore, the Riemann curvature tensor is antisymmetric on the first pair of indices,

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

From the explicit expression for $R^{\alpha}_{\ \beta\mu\nu}$, the curvature tensor must also be antisymmetric on the last pair of indices,

$$\begin{aligned} R^{\alpha}_{\ \beta\mu\nu} &= \Gamma^{\alpha}_{\ \beta\mu,\nu} - \Gamma^{\alpha}_{\ \beta\nu,\mu} + \Gamma^{\alpha}_{\ \sigma\nu}\Gamma^{\sigma}_{\ \beta\mu} - \Gamma^{\alpha}_{\ \sigma\mu}\Gamma^{\sigma}_{\ \beta\nu} \\ &= -\Gamma^{\alpha}_{\ \beta\nu,\mu} + \Gamma^{\alpha}_{\ \beta\mu,\nu} - \Gamma^{\alpha}_{\ \sigma\mu}\Gamma^{\sigma}_{\ \beta\nu} + \Gamma^{\alpha}_{\ \sigma\nu}\Gamma^{\sigma}_{\ \beta\mu} \\ &= -\left(\Gamma^{\alpha}_{\ \beta\nu,\mu} - \Gamma^{\alpha}_{\ \beta\mu,\nu} + \Gamma^{\alpha}_{\ \sigma\mu}\Gamma^{\sigma}_{\ \beta\nu} - \Gamma^{\alpha}_{\ \sigma\nu}\Gamma^{\sigma}_{\ \beta\mu}\right) \\ &= -R^{\alpha}_{\ \beta\nu\mu} \end{aligned}$$

Another symmetry follows if we totally antisymmetrize the final three indices,

$$\begin{aligned} R^{\alpha}_{\ [\beta\mu\nu]} &= \frac{1}{3}\left(R^{\alpha}_{\ \beta\mu\nu} + R^{\alpha}_{\ \mu\nu\beta} + R^{\alpha}_{\ \nu\beta\mu}\right) \\ 3R^{\alpha}_{\ [\beta\mu\nu]} &= \Gamma^{\alpha}_{\ \beta\mu,\nu} - \Gamma^{\alpha}_{\ \beta\nu,\mu} + \Gamma^{\alpha}_{\ \sigma\nu}\Gamma^{\sigma}_{\ \beta\mu} - \Gamma^{\alpha}_{\ \sigma\mu}\Gamma^{\sigma}_{\ \beta\nu} \\ &\quad + \Gamma^{\alpha}_{\ \mu,\nu\beta} - \Gamma^{\alpha}_{\ \mu\beta,\nu} + \Gamma^{\alpha}_{\ \sigma\beta}\Gamma^{\sigma}_{\ \mu\nu} - \Gamma^{\alpha}_{\ \sigma\nu}\Gamma^{\sigma}_{\ \mu\beta} \\ &\quad + \Gamma^{\alpha}_{\ \nu,\beta\mu} - \Gamma^{\alpha}_{\ \nu\mu,\beta} + \Gamma^{\alpha}_{\ \sigma\mu}\Gamma^{\sigma}_{\ \nu\beta} - \Gamma^{\alpha}_{\ \sigma\beta}\Gamma^{\sigma}_{\ \nu\mu} \\ &= \left(\Gamma^{\alpha}_{\ \beta\mu} - \Gamma^{\alpha}_{\ \mu\beta}\right)_{,\nu} + \left(\Gamma^{\alpha}_{\ \nu\beta} - \Gamma^{\alpha}_{\ \beta\nu}\right)_{,\mu} + \left(\Gamma^{\alpha}_{\ \mu\nu} - \Gamma^{\alpha}_{\ \nu\mu}\right)_{,\beta} \\ &\quad + \Gamma^{\alpha}_{\ \sigma\nu}\left(\Gamma^{\sigma}_{\ \beta\mu} - \Gamma^{\sigma}_{\ \mu\beta}\right) + \Gamma^{\alpha}_{\ \sigma\beta}\left(\Gamma^{\sigma}_{\ \mu\nu} - \Gamma^{\sigma}_{\ \nu\mu}\right) \\ &\quad + \Gamma^{\alpha}_{\ \sigma\mu}\left(\Gamma^{\sigma}_{\ \nu\beta} - \Gamma^{\sigma}_{\ \beta\nu}\right) \\ &= 0 \end{aligned}$$

the sum vanishes identically because of the symmetry of $\Gamma^{\alpha}_{\ \mu\nu}$. This condition ultimately arises because the connection may be written in terms of the metric. It is called the first Bianchi identity.

Finally, consider

$$\begin{aligned} R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} &= R_{\alpha\beta\mu\nu} - \left(-R_{\mu\alpha\beta\nu} - R_{\mu\beta\nu\alpha}\right) \\ &= R_{\alpha\beta\mu\nu} - R_{\alpha\mu\beta\nu} - R_{\beta\mu\nu\alpha} \\ &= R_{\alpha\beta\mu\nu} - \left(-R_{\alpha\beta\nu\mu} - R_{\alpha\nu\mu\beta}\right) - \left(-R_{\beta\alpha\mu\nu} - R_{\beta\nu\alpha\mu}\right) \\ &= R_{\alpha\beta\mu\nu} + R_{\alpha\beta\nu\mu} + R_{\alpha\nu\mu\beta} + R_{\beta\alpha\mu\nu} + R_{\beta\nu\alpha\mu} \\ &= -R_{\alpha\beta\mu\nu} + R_{\alpha\nu\mu\beta} + R_{\beta\nu\alpha\mu} \\ &= \left(-R_{\alpha\beta\mu\nu} - R_{\alpha\nu\beta\mu}\right) + R_{\beta\nu\alpha\mu} \\ &= R_{\alpha\mu\nu\beta} + R_{\beta\nu\alpha\mu} \\ &= -R_{\alpha\mu\beta\nu} + R_{\beta\nu\alpha\mu} \end{aligned}$$

Now interchange the names of both $\alpha\beta$ and $\mu\nu$. On the left, this gives two minus signs, but on the right only one:

$$R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = -R_{\alpha\mu\beta\nu} + R_{\beta\nu\alpha\mu}$$

$$\begin{aligned} R_{\beta\alpha\nu\mu} - R_{\nu\mu\beta\alpha} &= -R_{\beta\nu\alpha\mu} + R_{\alpha\mu\beta\nu} \\ (-1)^2\left(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}\right) &= +R_{\alpha\mu\beta\nu} - R_{\beta\nu\alpha\mu} \\ &= -\left(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}\right) \end{aligned}$$

This difference therefore vanishes, and we have symmetry under interchange of the pairs,

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

Summarizing, we have the following symmetries of the Riemann curvature tensor:

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\ R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\ R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\ R_{\alpha[\beta\mu\nu]} &= 0 \end{aligned}$$

We can count the independent components by using these symmetries. Because of the antisymmetry on $\alpha\beta$, there are only $\frac{4\cdot3}{2} = 6$ independent values for this pair of indices. The same counting holds for the final pair, $\mu\nu$. Since we have symmetry in these pairs,

$$R_{[\alpha\beta][\mu\nu]} = R_{[\mu\nu][\alpha\beta]}$$

we may think of $R_{[\alpha\beta][\mu\nu]}$ as a 6×6 symmetric matrix, which will have $\frac{6\cdot7}{2} = 21$ independent components. This makes use of the first three symmetries.

To use the final symmetry, note that the three final indices must differ from one another, so there are only possible four cases,

$$\begin{aligned} R_{0[\alpha\beta\mu]} &= 0 \\ R_{1[\alpha\beta\mu]} &= 0 \\ R_{2[\alpha\beta\mu]} &= 0 \\ R_{3[\alpha\beta\mu]} &= 0 \end{aligned}$$

Now suppose one of $\alpha\beta\mu$ is the same as the first index, for example,

$$\begin{aligned} R_{1123} + R_{1312} + R_{1231} &= R_{1312} + R_{1231} \\ &= R_{1312} - R_{1213} \\ &= 0 \end{aligned}$$

Then the vanishing is automatic using the previous three symmetries and there is no additional constraint. Therefore, to get any new condition, all four indices must differ. But then, notice that

$$\begin{aligned} R_{0123} &= -R_{1023} \\ &= R_{2310} \\ &= -R_{3210} \end{aligned}$$

so that once we have the condition with 0 in the first position, the other three possibilities follow automatically. There is therefore only one condition from the fourth symmetry,

$$R_{0123} + R_{0312} + R_{0231} = 0$$

reducing the number of degrees of freedom of the Riemann curvature tensor to 20.

3 Ricci tensor and Ricci scalar

Because of the symmetries, there is only one independent contraction of $R^{\alpha}{}_{\beta\mu\nu}$. We define the Ricci tensor,

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$$

Because of the symmetry between pairs, we have

$$\begin{aligned}
R_{\mu\nu} &\equiv R^{\alpha}_{\ \mu\alpha\nu} \\
&= g^{\alpha\beta}R_{\alpha\mu\beta\nu} \\
&= g^{\alpha\beta}R_{\beta\nu\alpha\mu} \\
&= R^{\alpha}_{\ \nu\alpha\mu} \\
&= R_{\nu\mu}
\end{aligned}$$

so the Ricci tensor is symmetric.

Exercise: Prove that any other contraction of the curvature is either zero, or a multiple of the Ricci tensor.

We also define the Ricci scalar, given by taking contracting the Ricci tensor,

$$R = g^{\mu\nu}R_{\mu\nu}$$

It is possible to decompose the full Riemann curvature into a traceless part, called the Weyl curvature, and combinations of the Ricci tensor and Ricci scalar, but we will not need this now.

4 The second Bianchi identity and the Einstein equation

We have already seen the first Bianchi identity,

$$R_{\alpha[\beta\mu\nu]} = 0$$

This is an integrability condition that guarantees that the connection may be written in terms of derivatives of the metric. There is a second integrability condition, called the second Bianchi identity, guaranteeing that the curvature may be written in terms of a connection. The second Bianchi identity is

$$R^{\alpha}_{\ \beta[\mu\nu;\sigma]} = 0$$

To prove this, first consider antisymmetrizing a double commutator:

$$\begin{aligned}
[D_\beta, [D_\mu, D_\nu]]\, w^\alpha &= (D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu - D_\mu D_\nu D_\beta + D_\nu D_\mu D_\beta)\, w^\alpha \\
3\left[D_{[\beta}, \left[D_\mu, D_{\nu]}\right]\right] &= [D_\beta, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\beta]] + [D_\nu, [D_\beta, D_\mu]] \\
&= D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu - D_\mu D_\nu D_\beta + D_\nu D_\mu D_\beta \\
&\quad + D_\mu D_\nu D_\beta - D_\mu D_\beta D_\nu - D_\nu D_\beta D_\mu + D_\beta D_\nu D_\mu \\
&\quad + D_\nu D_\beta D_\mu - D_\nu D_\mu D_\beta - D_\beta D_\mu D_\nu + D_\mu D_\beta D_\nu \\
&= D_\beta D_\mu D_\nu - D_\beta D_\mu D_\nu - D_\beta D_\nu D_\mu + D_\beta D_\nu D_\mu \\
&\quad + D_\mu D_\nu D_\beta - D_\mu D_\nu D_\beta - D_\mu D_\beta D_\nu + D_\mu D_\beta D_\nu \\
&\quad - D_\nu D_\beta D_\mu + D_\nu D_\beta D_\mu - D_\nu D_\mu D_\beta + D_\nu D_\mu D_\beta \\
&= 0
\end{aligned}$$

However, we may also write this as

$$\begin{aligned}
0 &= 3\left[D_{[\beta}, \left[D_\mu, D_{\nu]}\right]\right] w^\alpha \\
&= [D_\beta, [D_\mu, D_\nu]]\, w^\alpha + [D_\mu, [D_\nu, D_\beta]]\, w^\alpha + [D_\nu, [D_\beta, D_\mu]]\, w^\alpha \\
&= D_\beta\,(D_\mu D_\nu - D_\nu D_\mu)\, w^\alpha + D_\beta\,(D_\nu D_\mu - D_\mu D_\nu)\, w^\alpha \\
&\quad + D_\mu\,(D_\nu D_\beta - D_\beta D_\nu)\, w^\alpha + D_\mu\,(D_\beta D_\nu - D_\nu D_\beta)\, w^\alpha
\end{aligned}$$

$$
\begin{aligned}
&+D_\nu\left(D_\mu D_\beta - D_\beta D_\mu\right) w^\alpha + D_\nu\left(D_\beta D_\mu - D_\mu D_\beta\right) w^\alpha \\
= \; & D_\beta\left(w^\rho R^\alpha_{\ \rho\mu\nu}\right) + D_\beta\left(w^\rho R^\alpha_{\ \rho\nu\mu}\right) + D_\mu\left(w^\rho R^\alpha_{\ \rho\nu\beta}\right) \\
&+D_\mu\left(w^\rho R^\alpha_{\ \rho\beta\nu}\right) + D_\nu\left(w^\rho R^\alpha_{\ \rho\mu\beta}\right) + D_\nu\left(w^\rho R^\alpha_{\ \rho\beta\mu}\right) \\
= \; & w^\rho R^\alpha_{\ \rho[\mu\nu;\beta]} \\
&+\left(R^\alpha_{\ \rho\mu\nu} + R^\alpha_{\ \rho\nu\mu}\right) D_\beta w^\rho + \left(R^\alpha_{\ \rho\nu\beta} + R^\alpha_{\ \rho\beta\nu}\right) D_\mu w^\rho + \left(R^\alpha_{\ \rho\mu\beta} + R^\alpha_{\ \rho\beta\mu}\right) D_\nu w^\rho \\
= \; & w^\rho R^\alpha_{\ \rho[\mu\nu;\beta]}
\end{aligned}
$$

and since w^α is arbitrary, we have the second Bianchi identity.

The second Bianchi identity is important for general relativity because of its contractions. First, expand the identity and contract on $\alpha\mu$,

$$
\begin{aligned}
0 &= R^\alpha_{\ \beta\mu\nu;\sigma} + R^\alpha_{\ \beta\nu\sigma;\mu} + R^\alpha_{\ \beta\sigma\mu;\nu} \\
0 &= R^\alpha_{\ \beta\alpha\nu;\sigma} + R^\alpha_{\ \beta\nu\sigma;\alpha} + R^\alpha_{\ \beta\sigma\alpha;\nu}
\end{aligned}
$$

Using the definition of the Ricci tensor and the antisymmetry of the Riemann tensor,

$$
0 = R_{\beta\nu;\sigma} + R^\alpha_{\ \beta\nu\sigma;\alpha} - R_{\beta\sigma;\nu}
$$

Now contract on $\beta\sigma$ using the metric,

$$
\begin{aligned}
0 &= g^{\beta\sigma} R_{\beta\nu;\sigma} + g^{\beta\sigma} R^\alpha_{\ \beta\nu\sigma;\alpha} - g^{\beta\sigma} R_{\beta\sigma;\nu} \\
&= \left(g^{\beta\sigma} R_{\beta\nu}\right)_{;\sigma} + g^{\beta\sigma} R_{\beta\ \sigma\nu;\alpha}^{\ \alpha} - \left(g^{\beta\sigma} R_{\beta\sigma}\right)_{;\nu} \\
&= R^\sigma_{\ \nu;\sigma} + R^\alpha_{\ \nu;\alpha} - R_{;\nu}
\end{aligned}
$$

We may write this as a divergence,

$$
\begin{aligned}
0 &= R^\alpha_{\ \nu;\alpha} - \frac{1}{2} R_{;\nu} \\
0 &= D_\alpha\left(R^\alpha_{\ \nu} - \frac{1}{2}\delta^\alpha_\nu R\right)
\end{aligned}
$$

or, raising an index,

$$
D_\alpha\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right) = 0
$$

We define the Einstein tensor,

$$
G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R
$$

and have now shown that it has vanishing divergence. Since both the Ricci tensor and the metric are both symmetric, we have

$$
\begin{aligned}
G^{\alpha\beta} &= G^{\beta\alpha} \\
D_\beta G^{\alpha\beta} &= 0
\end{aligned}
$$

These are precisely the properties we require of the energy-momentum tensor, $T^{\alpha\beta}$. It can be shown that $G^{\alpha\beta}$ is the only tensor linear in components of the Riemann curvature tensor to have these properties.

Reasoning that it is the presence of energy that leads to curvature, the only candidate equation consistent with the properties of $T^{\alpha\beta}$ and linear in the curvature (hence, a second order differential equation for the metric) is

$$
G^{\alpha\beta} = \kappa T^{\alpha\beta}
$$

This is the Einstein equation for general relativity.