Introduction to Gauge Theory

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Outline of Lecture

Introduction

Schrödinger-Maxwell Theory

A Review of Differential Geometry

Gauge Theory on Open Sets

The inventors of gauge theory

Figure : James Clerk Maxwell and Hermann Weyl

A fruitful failure

1919. № 10.

ANNALEN DER PHYSIK.
VIERTE FOLGE. BAND 59.

1. *Eine neue Erweiterung der Relativitätstheorie;*
von H. Weyl.

Kap. I. Geometrische Grundlage.

Einleitung. Um den physikalischen Zustand der Welt an einer Weltstelle durch Zahlen charakterisieren zu können, muß 1. die Umgebung dieser Stelle auf *Koordinaten* bezogen sein und müssen 2. gewisse *Maßeinheiten* festgelegt werden. Die bisherige Einsteinsche Relativitätstheorie bezieht sich nur auf den ersten Punkt, die Willkürlichkeit des Koordinatensystems; doch gilt es, eine ebenso prinzipielle Stellungnahme zu dem zweiten Punkt, der Willkürlichkeit der Maßeinheiten, zu gewinnen. Davon soll im folgenden die Rede sein.

Die Welt ist ein vierdimensionales Kontinuum und läßt sich deshalb auf vier Koordinaten $x_0\, x_1\, x_2\, x_3$ beziehen. Der Übergang zu einem anderen Koordinatensystem $\bar{x}_i$ wird durch stetige Transformationsformeln

(1) $$x_i = f_i(\bar{x}_0\, \bar{x}_1\, \bar{x}_2\, \bar{x}_3) \qquad (i = 0, 1, 2, 3)$$

vermittelt. An sich ist unter den verschiedenen möglichen Koordinatensystemen keines ausgezeichnet. Die Relativkoordinaten $d x_i$ eines zu dem Punkte $P = (x_i)$ unendlich benachbarten $P' = (x_i + d x_i)$ sind die Komponenten der infinitesimalen Verschiebung $\overrightarrow{P P'}$ (eines „Linienelementes" in P). Sie transformieren sich beim Übergang (1) zu einem anderen Koordinatensystem $\bar{x}_i$ linear:

(2) $$d x_i = \sum_k \alpha_k^i \, d \bar{x}_k ;$$

α_k^i sind die Werte der Ableitungen $\partial f_i / \partial \bar{x}_k$ im Punkte P. In der gleichen Weise transformieren sich die Komponenten ξ^i irgendeines *Vektors* in P. Mit einem die Umgebung von P

Length scale ('gauge') depends on spacetime?

Parallel transport of a length scale $\ell \in \mathbb{R}$ in terms of 1-form $A = A_t dt + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$:

$$d\ell = -A\ell, \tag{1}$$

Change the gauge $\ell' = \lambda\ell, \qquad \lambda : \mathbb{R}^4 \to \mathbb{R}^+$.
Then

$$d\ell' = \lambda d\ell + d\lambda\ell = (-\lambda A + d\lambda)\ell = \lambda(-A + d\ln\lambda)\ell.$$

In order to maintain the condition (1) in the new gauge we require

$$A' = A - d\ln\lambda.$$

$F = dA$ is unchanged!
Electromagnetic field? Einstein: ruled out by experiment

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- Gauge theories now used in physics, mathematics, economics and finance.
- Here: gauge theory of ‘internal degrees of freedom’ (compact Lie groups)
- The unreasonable effectiveness of gauge theories in modern physics and mathematics. Why?

Outline of Lecture

Schrödinger Equation

Wavefunction of free particle non-relativistic quantum mechanics is a map $\psi : \mathbb{R}^4 \to \mathbb{C}$ which obeys

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi. \tag{2}$$

Normalised $\int_{\mathbb{R}^3} |\psi(t, \boldsymbol{x})|^2 d^3\boldsymbol{x} = 1, \quad \forall t \in \mathbb{R}$, so that the probability of the particle being in a region $R \subset \mathbb{R}^3$ at time t is

$$p(t, R) = \int_R |\psi(t, \boldsymbol{x})|^2 d^3\boldsymbol{x}.$$

The probability is invariant under a 'phase change'

$$\psi \mapsto \psi' = e^{i\chi}\psi, \quad \chi : \mathbb{R}^4 \to [0, 2\pi).$$

Modifying the Schrödinger Equation

Introduce functions a_t, a_1, a_2, a_3 on $\mathbb{R}^4$ and

$$D_t = \partial_t + ia_t, \quad D_j = \partial_j + ia_j, \quad j = 1, 2, 3.$$

Then

$$i\hbar D_t \psi = -\frac{\hbar^2}{2m} \sum_{j=1}^{3} D_j^2 \psi, \tag{3}$$

respects local phase rotations if we also map

$$a_t \mapsto a_t' = a_t - \partial_t \chi, \quad a_j \mapsto a_j' = a_j - \partial_j \chi.$$

Reason: $D_t' \psi' = e^{i\chi} D_t \psi$

Gauging the Schrödinger Equation

Introduce **gauge potential** a on $\mathbb{R}^4$ and **covariant derivatives**

$$D_t = \partial_t + ia_t, \quad D_j = \partial_j + ia_j, \quad j = 1, 2, 3.$$

Then the **minimally coupled/gauged** Schrödinger equation

$$i\hbar D_t \psi = -\frac{\hbar^2}{2m} \sum_{j=1}^{3} D_j^2 \psi, \tag{4}$$

is **covariant** if we apply **gauge transformation**

$$a_t \mapsto a_t' = a_t - \partial_t \chi, \quad a_j \mapsto a_j' = a_j - \partial_j \chi.$$

Reason: **Covariance** $D_t' \psi' = e^{i\chi} D_t \psi$

Maxwell theory from gauging

In vector field notation, $a_t \mapsto a_t' = a_t - \partial_t \chi, \qquad \boldsymbol{a} \mapsto \boldsymbol{a} - \nabla \chi,$
leaves invariant

$$\boldsymbol{e} = \nabla a_t - \partial_t \boldsymbol{a}, \qquad \boldsymbol{b} = \nabla \times \boldsymbol{a}.$$

Electric and magentic field? Yes:

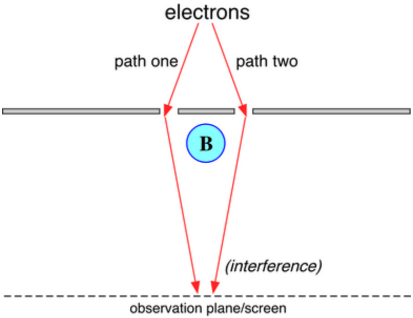
$$\nabla \cdot \boldsymbol{b} = 0, \qquad \nabla \times \boldsymbol{e} + \partial_t \boldsymbol{b} = 0 :$$

Homogeneous Maxwell equations: **'no magnetic monopoles'** and **Faraday's law of induction.** With electric charge density ρ and a current density $\boldsymbol{j}$:

$$\nabla \cdot \boldsymbol{e} = \rho, \qquad \nabla \times \boldsymbol{b} - \partial_t \boldsymbol{e} = \boldsymbol{j}.$$

Gauss and **Ampère-Maxwell** $\Leftarrow$ Schrödinger-Maxwell action.

Einstein's objection revisited: Aharonov-Bohm



Outline of Lecture

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Schrödinger-Maxwell Theory

A Review of Differential Geometry

Gauge Theory on Open Sets

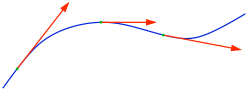
Notation

1. $U \subset \mathbb{R}^n$ an open set
2. $x = (x_1, \ldots, x_n)$ Cartesian coordinates on $U \subset \mathbb{R}^n$
3. $u = (u_1, \ldots, u_n)$ arbitrary coordinates - e.g. polar, cylindrical, ...
4. (t, x_1, x_2, x_3) for Cartesian coordinates on $\mathbb{R}^4$ interpreted as spacetime
5. $\partial_i = \frac{\partial}{\partial x_i}$ etc
6. V vector space

Vector field = directional derivative

Definition

For each $p \in U$, the tangent space T_pU to U at p is the set of velocities of all curves that pass through p. An element of T_pU is called a tangent vector and if c is a curve that passes through p, we let $c'(0)$ denote the corresponding element in T_pU.



Think of vector fields on $U \in \mathbb{R}^n$ as directional derivatives and write the action of a vector field v on a function $f \in C^\infty(U)$ as $v[f]$:

$$v = 2x_1\partial_1 - 3x_1x_2x_3\partial_2, \quad v[f] = 2x_1\partial_1 f - 3x_1x_2x_3\partial_2 f$$

Differential 1-forms

Recall dual vector space = space of linear maps $V \to \mathbb{R}$

Definition

A *1-form at* $p \in U \subset \mathbb{R}^n$ is a linear map $T_p(U) \to \mathbb{R}$. A *differential 1-form on* $U \subset \mathbb{R}^n$ is a smooth choice for each $p \in U$ of a 1-form at p. If α is a 1-form and v a vector field, we write $\alpha(v)$ for the function which is the result of applying α to v.

$$dx_j(\partial_k) = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha = 3dx_1 + 2x_3dx_2 - dx_3$ then

$$\alpha(x_2\partial_1) = 3x_2$$

Exterior Derivative

Definition

Given a function f on U, its exterior derivative is the 1-form df defined by

$$(df)(v) = v[f].$$

It is easy to check in coordinates that

$$df = \sum_{i=1}^{n} \partial_i f \, dx_i.$$

Differential k-forms

Definition

A *k-form at $p \in U \subset \mathbb{R}^n$* is a map $T_pU \times \cdots \times T_pU \to \mathbb{R}$ which is

- multilinear, i.e., linear in each of its k arguments,
- antisymmetric in any two of its arguments.

A *differential 1-form on $U \subset \mathbb{R}^n$* is a smooth choice for each $p \in U$ of a k-form at p. The first argument of a k-form denotes the dependence on the point. Write $\Omega^k(U)$ for the vector space of k-forms.

The integer k is called the degree of the form. A zero-form is a function on U.
Important extension: Write $\Omega^k(U, V)$ for V-valued k-forms.

Wedge Product and Exterior Derivatives of k-forms

With

$$\alpha = x_3 dx_1 \wedge dx_2, \quad v_1 = x_2\partial_1 + \partial_3, \quad v_2 = x_2\partial_2 + x_1\partial_1$$

we have

$$\alpha(v_1, v_2) = x_3(dx_1(v_1)dx_2(v_2) - dx_1(v_2)dx_2(v_1)) = x_3(x_2^2 - x_1).$$

Also

$$d(\sin(x_3)dx_1 \wedge dx_2) = \cos(x_3)dx_3 \wedge dx_1 \wedge dx_2 = \cos(x_3)dx_1 \wedge dx_2 \wedge dx_3.$$

Some Important Rules

(1) $dd\alpha = 0 \quad \forall \quad \alpha \in \Omega^k(U)$. **(Schwarz!)**

(2) $\alpha \wedge \beta = (-1)^{\deg \alpha \deg \beta} \beta \wedge \alpha$.

(3) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$.

...and some terminology:

Definition

A k-form α is called **closed** if $d\alpha = 0$. It is called **exact** if there exists a $(k-1)$-form β so that $\alpha = d\beta$

Exact forms are necessarily closed.

Integration of Forms over Open Sets

To integrate k-form over $U \subset \mathbb{R}^k$, need an orientation:

Definition

An orientation of $U \in \mathbb{R}^k$ is determined by a nowhere vanishing k-form ω on U. A coordinate system $u_1, \ldots, u_k$ is oriented if the associated basis $\{\partial_1, \ldots, \partial_k\}$ of the tangent spaces T_pU satisfies

$$\omega(p, \frac{\partial}{\partial u_1}, \ldots \frac{\partial}{\partial u_k}) > 0 \quad \forall p \in U.$$

E.g. orientation on $\mathbb{R}^2$ is given by $dx_1 \wedge dx_2$
Integrate a k-form α over an open set U with orientation:

$$\int_U \alpha = \int_U \alpha\left(u, \frac{\partial}{\partial u_1}, \ldots \frac{\partial}{\partial u_k}\right) du_1 \ldots du_k.$$

Independent of coordinate choice!

Integration over Submanifolds

Assuming the submanifold $R \subset \mathbb{R}^N$ is parametrised by an open set $U \in \mathbb{R}^k$ with coordinate $u = (u_1, \ldots, u_k)$ and by a map $\gamma : U \to \mathbb{R}^n$, we define the integral by 'pulling back' the form to the parameter space:

$$\int_R \alpha = \int \alpha \left(\gamma(u), \frac{\partial \gamma}{\partial u_1}, \ldots \frac{\partial \gamma}{\partial u_k} \right) du_1 \ldots du_k.$$

Stokes' Theorem

Theorem

Let R be a compact, oriented k-dimensional submanifold of $\mathbb{R}^n$ and $\alpha \in \Omega^{k-1}(U)$ for an open set $U \subset \mathbb{R}^n$ containing R. Denote the boundary of R with the induced orientation by ∂R. Then

$$\int_R d\alpha = \int_{\partial R} \alpha.$$

$k = 3$: Gauss' divergence theorem $\int_R \nabla \cdot \boldsymbol{E} = \int_{\partial R} \boldsymbol{E} \cdot \boldsymbol{n} dA$
$k = 2$: Stokes' theorem $\int_R (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{n} dA = \int_{\partial R} \boldsymbol{E} \cdot d\ell$

Metrics

Definition

If V is a real vector space, a metric η is a bilinear map $V \times V \to \mathbb{R}$ which is

1. symmmetric, i.e.,$\forall v, w \in V,\ \eta(v,w) = \eta(w,v)$,
2. non-degenerate, i.e., $\forall v,\ \eta(v,w) = 0 \Rightarrow w = 0$

A metric on an open set $U \subset \mathbb{R}^n$ is a smooth choice of metrics on T_pU for each $p \in U$.

Isomorphism $v \in V \mapsto \eta(v,\cdot) \in V^* \Rightarrow$ Induces a metric on V^* and tensor powers of V^*!

Minkowski space is $\mathbb{R}^4$ equipped with the metric $\eta = dt^2 - dx_1^2 - dx_2^2 - dx_3^2$

The Hodge Star Operator

Definition

With orientation ω of U, the Hodge star operator $\star : \Omega^k(U) \to \Omega^{n-k}(U)$ is defined via

$$\alpha \wedge (\star\beta) = \eta(\alpha, \beta)\, \omega \quad \forall \alpha, \beta \in \Omega^k(U).$$

In terms of an orthonormal and oriented basis $\{e_1, \ldots, e_n\}$ of 1-forms:

$$\star(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) = \eta_{i_1 i_1} \ldots \eta_{i_k i_k} e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \cdots \wedge e_{i_n},$$

where $\{i_1, \cdots i_k, i_{k+1} \cdots i_n\}$ is an even permutation of $\{1, 2, \cdots n\}$ and $\eta_{ij} = \eta(e_j, e_k)$

Self-dual Forms

Note that $\star\star\alpha = (-1)^{k(n-k)+s}\alpha, \quad \forall\alpha \in \Omega^k(U)$

Definition

A k-form α is called self-dual if $\star\alpha = \alpha$. It is called anti-self-dual $\star\alpha = -\alpha$

Example

On Minkowski space, with orientation $\omega = dt \wedge dx_1 \wedge dx_2 \wedge dx_3$

$$\star(dt \wedge dx_1) = -dx_2 \wedge dx_3, \star(dx_2 \wedge dx_3) = dt \wedge dx_1$$

Outline of Lecture

Maxwell Electrodynamics Revisited

Assemble gauge potentials a_t, a_1, a_2, a_3 into a 1-form on $\mathbb{R}^4$,

$$a = a_t dt + a_1 dx_1 + a_2 dx_2 + a_3 dx_3,$$

to find electromagnetic field strength

$$f = da = -\sum_{i=1}^{3} e_i dt \wedge dx_i + b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2,$$

Further defining a current 3-form as

$$j = \rho dx_1 \wedge dx_2 \wedge dx_3 - j_1 dt \wedge dx_2 \wedge dx_3 - j_2 dt \wedge dx_3 \wedge dx_1 - j_3 dt \wedge dx_1 \wedge dx_2,$$

Maxwell's equations are $\quad df = 0, \quad d \star f = j.$

Maxwell Electrodynamics is ...

1. First gauge theory
2. First special relativistic field theory
3. Prototype for Standard Model
4. First arena of ‘duality’ in physics

Ingredients of Schrödinger-Maxwell Theory

1. The 'phase rotation' $e^{i\chi}$, with $\chi \in [0, 2\pi)$,
2. The wavefunction $\psi \in \Omega^0(\mathbb{R}^4, \mathbb{C})$,
3. The gauge field $a \in \Omega^1(\mathbb{R}^4)$,
4. The modified derivative $d + ia$ of ψ,
5. The field strength $f = da \in \Omega^2(\mathbb{R}^4)$,
6. Local gauge changes $e^{i\chi}$, for $\chi : \mathbb{R}^4 \to [0, 2\pi)$ on ψ and a.

Comparing Schrödinger-Maxwell and Weyl's theory

- Phase rotation $e^{i\chi}$ are elements of $U(1)$. Weyl's scaling group $\mathbb{R}^+$ is also a Lie group,
- The Lie algebra (tangent space at the identity) of $U(1)$ is spanned by $\{i\}$. Interpret $A = ia$ as a Lie-algebra valued 1-form, so $D_A = d + A$
- Acting on a k-form,

$$(d+A)^2\alpha = (d+A)(d\alpha + A\wedge\alpha) = dA\wedge\alpha - A\wedge d\alpha + A\wedge d\alpha$$

Writing $F = dA$ or $F = if$, we thus have $D_A^2\alpha = F\wedge\alpha$.

Gauge ingredients

In order to define a gauge theory on $U \subset \mathbb{R}^n$ we require, in general,

1. A Lie group G, whose Lie algebra we denote by $\mathfrak{g}$.
2. A representation ρ of G on a vector space V. We denote the associated representation of $\mathfrak{g}$ on V by ρ_*.

Gauge recipe

1. Gauge changes at a point $p \in U$ are implemented by elements of G: *gauge group*
2. 'Wavefunctions' become local sections: $\Omega^k(U, V)$.
3. A local gauge field $A \in \Omega^1(U, \mathfrak{g})$,
4. The covariant exterior derivative $D_A := d + A$ acts on local sections

$$D_A : \Omega^k(U, V) \to \Omega^{k+1}(U, V), \quad \phi \mapsto d\phi + \rho_*(A) \wedge \phi.$$

5. The curvature/field strength is defined via the covariant derivative

$$(D_A)^2\phi = \rho_*(F_A) \wedge \phi.$$

6. Local changes of gauge $\gamma : U \to G$ act $A \mapsto A' = \gamma A \gamma^{-1} + \gamma d\gamma^{-1}$ and $\phi \mapsto \phi' = \rho(\gamma)\phi$.

Working with vector-valued forms

Expand $A \in \Omega^1(U, \mathfrak{g})$ as $A = \sum_{\alpha=1}^{d} A_\alpha t_\alpha$, for *ordinary* 1-forms A_α. Then,

$$[A, A] = \sum_{\alpha,\beta=1}^{d} A_\alpha \wedge A_\beta [t_\alpha, t_\beta].$$

Applying the representation ρ_* we note the useful rule

$$\begin{aligned} \rho_*([A, A]) &= \sum_{\alpha,\beta=1}^{d} A_\alpha \wedge A_\beta [\rho_*(t_\alpha), \rho_*(t_\beta)] \\ &= \sum_{\alpha,\beta=1}^{d} A_\alpha \wedge A_\beta (\rho_*(t_\alpha)\rho_*(t_\beta) - \rho_*(t_\beta)\rho_*(t_\alpha)) \\ &= 2\rho_*(A) \wedge \rho_*(A). \end{aligned} \tag{5}$$

Curvature and Gauge Transformations

$$\begin{aligned}
&(D_A)^2\phi \\
&= (d + \rho_*(A))(d\phi + \rho_*(A) \wedge \phi) \\
&= \rho_*(A) \wedge d\phi + \rho_*(dA) \wedge \phi - \rho_*(A) \wedge d\phi + \rho_*(A) \wedge \rho_*(A) \wedge \phi \\
&= \rho_*(dA) \wedge \phi + \frac{1}{2}\rho_*([A, A]) \wedge \phi, \qquad (6)
\end{aligned}$$

Thus

$$F_A = dA + \frac{1}{2}[A \wedge A]$$

or $F_A = dA + A \wedge A$ for matrix groups.
Under gauge change $F_{A'} = \gamma F_A \gamma^{-1}$, and $D_{A'}\phi' = \rho(\gamma) D_A \phi$.

Gauge Theories in Physics

Write down gauge-invariant *action*

$$S = -\int_U \kappa(F_A \wedge \star F_A) + \mathrm{tr}\,(D_A\phi \wedge \star D_A\phi) + W(\phi)$$

where κ is the (negative definite) Killing form on $\mathfrak{g}$ and $W : V \to \mathbb{R}$ is a *potential*.
Require G-invariance $W(\rho(g)(\phi)) = W(\phi)$
'Mexican hat potential' for a field ϕ in the defining representation of a matrix group G:

$$W(\phi) = (\lambda + \mathrm{tr}\,(\phi^2))^2, \tag{7}$$

where λ is a parameter which sets a mass scale in the theory via the **Higgs effect**.